Oscillatory Integrals and Eigenfunction Restriction Estimates

by
Cheng Zhang

A dissertation submitted to Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland
June 2019

Abstract

The thesis consists of two parts. In the first part, we prove optimal L^2 estimates of the restrictions to a family of curves of the eigenfunctions on general compact smooth 3-dimensional Riemannian manifolds. This family includes geodesics, smooth curves with nonvanishing geodesic curvatures, and those curves satisfying certain finite-type condition, such as all the analytic curves on analytic manifolds. These results sharpen the corresponding bounds of Burq, Gérard and Tzvetkov [5], Hu [12], Chen and Sogge [8]. We show that the problem is essentially related to Hilbert transforms along curves in the plane and a class of singular oscillatory integrals studied by Phong and Stein[17], Ricci and Stein[20], Pan[16], Seeger[21], Carbery and Pérez[6], Nagel and Wainger [15].

In the second part of the thesis, we show that one can obtain logarithmic improvements of L^2 geodesic restriction estimates for eigenfunctions on 3-dimensional compact Riemannian manifolds with constant negative curvature. We obtain a $(\log\lambda)^{-\frac{1}{2}}$ gain for the L^2-restriction bounds, which improves the corresponding bounds of Burq, Gérard and Tzvetkov [5], Hu [12], Chen and Sogge [8]. We achieve this by adapting the approaches developed by Chen and Sogge [8], Blair and Sogge [3], Xi and the author [27]. We derive an explicit formula for the wave kernel on 3D hyperbolic space, which improves the kernel estimates from the Hadamard parametrix in Chen and Sogge [8]. We prove detailed oscillatory integral estimates with fold singularities by Phong and Stein [18] and use the Poincaré half-space model to establish bounds for various derivatives of the distance function restricted to geodesic segments on the universal cover $\mathbb{H}^3$.

READERS: Professor Christopher D. Sogge (Advisor) and Professor Benjamin Dodson

Acknowledgments

First of all, I would like to thank my advisor Professor Christopher D. Sogge for his patient guidance and many helpful discussions during the 5 years at Hopkins. So here I want to express my deepest gratitude to him.

I would like to thank Steve Zelditch, Yannick Sire, Benjamin Dodson, Yi Wang and Richard Brown. In my heart, I am deeply grateful to them for their recommendation letters when I was applying for jobs. Moreover, I would like to express my gratitude to Joel Spruck, Chikako Mese and Steven Zucker.

I would like to thank my colleagues, collaborators, and friends in China and US, including Jian Kong, Sabrina Raymond, Xing Wang, Yakun Xi, Tianyi Ren, Xiaoqi Huang, Liuquan Wang, Jianxin Sun, Bin Ren, Ning Liu, and many others. They made my graduate life easier and more fun.

Finally, I would like to thank my parents in China. Their advice and encouragement are invaluable. I dedicate this dissertation to my parents.

Contents

List of Figures

1

Sharp eigenfunction restriction estimates and Hilbert transforms along curves

1.1 Introduction

Let (M,g) be a compact smooth n-dimensional Riemannian manifold, $n \geqslant 2$, and let Δ_g be the associated Laplace-Beltrami operator. Let e_λ denote the L^2-normalized eigenfunction

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

so that $\lambda \geqslant 0$ is the eigenvalue of the operator $\sqrt{-\Delta_g}$. For the classical results on $L^p(M)$ norms of e_λ, see Sogge [23]. For $2 \leqslant p \leqslant \infty$,

$$\|e_\lambda\|_{L^p(M)} \leqslant C\lambda^{\delta(p)},$$

where $\delta(p) = \max\{\frac{n-1}{2} - \frac{n}{p}, \frac{n-1}{4} - \frac{n-1}{2p}\}$.

For recent improvements on nonpositively curved manifolds, see Bérard [1], Hassell and Tacy [9], Hazari and Rivière [11], Blair and Sogge [3][4], Sogge [22].

L^p-estimates have been established for the restriction of eigenfunctions to submanifolds by Burq, Gérard and Tzvetkov [5], Hu [12](see also [19] for earlier results on hyperbolic surfaces). For $2 \leqslant$

$p \leqslant \infty$,

$$\|e_\lambda\|_{L^p(\Sigma)} \leqslant C\lambda^{\sigma(n,k,p)}(\log\lambda)^{\rho(n,k,p)}, \tag{1.1.1}$$

where Σ is a smooth submanifold of dimension k, $1 \leqslant k < n$,

$$\sigma(n,k,p) = \max\Big\{\frac{n-1}{2} - \frac{k}{p}, \frac{n-1}{4} - \frac{k-1}{2p}\Big\},$$

and one can take

$$\rho(n,k,p) = \begin{cases} \frac{1}{2} & (k,p) = (n-2,2), \\ 0 & \text{otherwise}. \end{cases}$$

For example, for $k = 1$,

$$\sigma(2,1,p) = \begin{cases} \frac{1}{4} & 2 \leqslant p \leqslant 4 \\ \frac{1}{2} - \frac{1}{p} & 4 < p \leqslant \infty \end{cases}, \quad \sigma(n,1,p) = \frac{n-1}{2} - \frac{1}{p},\ p \geqslant 2,\ n \geqslant 3.$$

Note that the exponents $\sigma(n,k,p)$ agree with Sogge's exponents $\delta(p)$ when $k = n$. All results are sharp on the sphere $\mathbb{S}^n$, except for the log loss.

For recent improvements about the restriction to submanifolds on nonpositively curved manifolds, see Chen [7], Chen and Sogge [8], Blair and Sogge [3], Xi and Zhang [27], Hezari [10], Blair [2], Zhang [28].

Here we limit the discussion for $n = 3$ and the restriction to the smooth curve $\gamma : [a,b] \to M$ parametrized by arc length. Burq, Gérard and Tzvetkov [5], Hu [12] showed that

$$\|e_\lambda\|_{L^p(\gamma)} \leqslant C_p\lambda^{1-\frac{1}{p}}\|e_\lambda\|_{L^2(M)},\ p > 2, \tag{1.1.2}$$

which are saturated by zonal functions on S^3. In the case of $p = 2$, they also obtained

$$\|e_\lambda\|_{L^2(\gamma)} \leqslant C\lambda^{\frac{1}{2}}(\log\lambda)^{\frac{1}{2}}\|e_\lambda\|_{L^2(M)}, \tag{1.1.3}$$

which are not saturated on S^3. Chen and Sogge [8] showed that the $\log\lambda$ can be removed if γ is a geodesic segment, by using the L^2-boundedness of Hilbert transform. So whether the log loss in (1.1.3) can be removed for general smooth curves on general manifolds is an interesting open problem.

The problem is related to the singular integral operators T_λ are of the form

$$T_\lambda f(t) = \text{p.v.} \int e^{i\lambda\phi(t,s)}(t-s)^{-1}a(t,s)f(s)ds, \tag{1.1.4}$$

where ϕ is smooth, λ is real, and $a \in C_0^\infty(\mathbb{R}^2)$. These operators and their generalizations in higher dimensions have been studied by Phong and Stein [17], Ricci and Stein [20], Pan [16], Seeger [21], Carbery and Pérez [6]. It was shown in [17, p.117] that uniform $L^2(\mathbb{R})$ estimates of T_λ can be applied to show $L^2(\mathbb{R}^2)$ boundedness of Hilbert transform $\mathcal{H}$ along variable curves:

$$\|\mathcal{H}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant \sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}.$$

Here $\mathcal{H}$ is defined a priori on functions in $C_0^\infty(\mathbb{R}^2)$:

$$\mathcal{H}f(x) = \eta(x)\text{p.v.} \int_{-\delta}^{\delta} f(x_1 - t, x_2 - \phi(x_1, x_1 - t))\frac{dt}{t},$$

where $\eta \in C_0^\infty(\mathbb{R}^2)$ and $\delta > 0$ is suitably small. Pan [16, Theorem 2] proved that T_λ is uniformly bounded on $L^2(\mathbb{R})$ if one imposes a weak finite type condition: the mixed derivative ϕ''_{ts} does not vanish of infinite order on supp a (e.g. the phase function ϕ is real-analytic). Later, Seeger [21], Carbery and Pérez [6] considered certain "flat" cases where the finite type condition is not satisfied. In the model case, $\phi(t,s) = \psi(t-s)$, Nagel, Vance, Wainger and Weinberg [14] proved necessary and sufficient conditions in the case that ψ is even(or odd) and convex. Nagel and Wainger [15, Theorem 4.1] found an odd smooth function $\psi(t)$ on $[-1,1]$, which vanishes of infinite order at $t = 0$, such that the Hilbert transform $\mathcal{H}$ along the curve $(t, \psi(t))$ is unbounded on $L^2(\mathbb{R}^2)$. This implies that the operators T_λ may not be uniformly bounded on $L^2(\mathbb{R})$ if the finite type condition in [16, Theorem 2] is removed.

In this chapter, we extend Chen-Sogge's results [8, Theorem 1] on geodesics to more general curves. Indeed, we show that the log-factor in (1.1.3) can be removed for a family of curves, including geodesics, smooth curves with nonvanishing geodesic curvatures, and those curves satisfying certain finite-type condition, such as all the analytic curves on analytic manifolds. These results are optimal and saturated on the sphere S^3. We use the wave kernel method and the Hadamard parametrix to reduce the problem to the uniform L^2-estimates of a class of singular integrals with oscillatory terms. We first give an independent proof for smooth curves with nonvanishing geodesic curvature, by using Hörmander's oscillatory integral theorem. For more general cases, we apply the oscillatory integral theorem in Pan [16] to the curves satisfying certain finite type condition. Throughout this chapter, the injective radius of M is sufficiently large, and using a partition of unity, we may assume that γ is a segment contained in the domain of a given coordinate patch.

Theorem 1. *Let (M, g) be a compact smooth 3-dimensional Riemannian manifold. Let $\gamma \subset M$ be a fixed unit-length curve with nonvanishing geodesic curvatures. Then for $\lambda \gg 1$, there is a constant C independent of λ such that*

$$\|e_\lambda\|_{L^2(\gamma)} \leqslant C\lambda^{\frac{1}{2}}\|e_\lambda\|_{L^2(M)}. \tag{1.1.5}$$

This result is sharp on the standard sphere S^3. See [5, p. 36] for the proof of sharpness. Suppose that the distance function on (M, g) is $d_g(\cdot,\cdot) : M \times M \to [0, +\infty)$. Let $\gamma : [-\frac{1}{2}, \frac{1}{2}] \to M$ be a smooth curve segment parametrized by are length. Using Taylor expansion, we can give a precise description of $d_g(\gamma(t), \gamma(s))$.

Lemma 1. *We can write for $t, s \in [-\frac{1}{2}, \frac{1}{2}]$,*

$$d_g(\gamma(t), \gamma(s)) = |t-s|(1 - c(t)(t-s)^2 + d(t, t-s)(t-s)^3), \tag{1.1.6}$$

where $c(t)$ and $d(t, t-s)$ are smooth functions. And $c(t) \geqslant c_0 > 0$ if γ has nonvanishing geodesic curvatures.

Proof. See Burq, Gérard and Tzvetkov [5, Lemma 4.5]. Their proof is for curves on surfaces, while the same argument works on general n-dimensional manifolds. $\square$

Thus, if we set $\phi(t, s) = d_g(\gamma(t), \gamma(s))\mathrm{sgn}(t-s)$ then $\phi(t, s) \in C^\infty[-\frac{1}{2}, \frac{1}{2}]^2$. In particular, if (M, g) and γ are both analytic, $\phi(t, s) \in C^\omega[-\frac{1}{2}, \frac{1}{2}]^2$. We say γ satisfies the **finite type condition** if the mixed derivative ϕ''_{ts} does not vanish of infinite order on $[-\frac{1}{2}, \frac{1}{2}]^2$.

Proposition 1 ([16])**.** *Suppose that T_λ is defined in (1.1.4), and ϕ''_{st} does not vanish of infinite order on* supp a*. Then the operators T_λ are uniformly bounded on L^p to itself, for $1 < p < \infty$.*

Corollary 1. *If the phase function ϕ is real analytic, then T_λ are uniformly bounded on L^p, for $1 < p < \infty$.*

The proof can be found in [16, page 212–216]. The basic idea of the proof of Proposition 1 is to break the operator T_λ into two parts, one supported near the origin, and the other away from the origin. To estimate the first part, one can use approximation to the phase function by a polynomial of sufficiently high order and then apply the Ricci-Stein theorem [20]. To estimate the second part, one can use the finite type condition to apply the Malgrange Preparation theorem. Note that the finite type condition is only used to estimate the second part.

Theorem 2. *If γ satisfies the finite type condition, then for $\lambda \gg 1$, there is a constant C such that*

$$\|e_\lambda\|_{L^2(\gamma)} \leqslant C\lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(M)}. \tag{1.1.7}$$

Corollary 2. *If (M, g) and γ are both analytic, then (1.1.7) holds.*

Remark 1. Theorem 1 is a special case of Theorem 2. One may also have (1.1.7) if the finite type condition is replaced by certain conditions in [21], [6]. It is interesting to consider whether (1.1.7) can hold for *any* smooth curves (or find a counterexample), and their generalizations in higher dimensions. Moreover, it is also interesting to know whether the operators T_λ are uniformly bounded on $L^2(\mathbb{R})$ if ϕ is the distance function $\phi(t, s) = d_g(\gamma(t), \gamma(s))\mathrm{sgn}(t - s)$ on *any* smooth curves γ (or find a counterexample). See [15, Theorem 4.1] for a counterexample with an explicit phase function $\phi(t, s) = \psi(t - s)$, but this phase function is not a distance function on a curve.

1.2 Proof of the theorems

1.2.1 Preliminaries

We start with some standard reductions. Let $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = 1$ and supp $\hat{\rho} \subset [-1/2, 1/2]$, then it is clear that the operator $\rho(\lambda - \sqrt{-\Delta_g})$ reproduces eigenfunctions, namely

$$\rho(\lambda - \sqrt{-\Delta_g})e_\lambda = e_\lambda.$$

Let $\chi = |\rho|^2$. After a standard TT^* argument, we only need to estimate the norm

$$\|\chi(\lambda - \sqrt{-\Delta_g})\|_{L^2(\gamma)\to L^2(\gamma)}. \tag{1.2.1}$$

We rewrite the kernel $\chi(\lambda - \sqrt{-\Delta_g})(x, y)$ by the Fourier inversion formula

$$\begin{aligned}\chi(\lambda - \sqrt{-\Delta_g})(x, y) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\chi}(\tau)e^{i\lambda\tau}(e^{-i\tau\sqrt{-\Delta_g}})(x, y)d\tau \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \hat{\chi}(\tau)e^{i\lambda\tau}\cos(\tau\sqrt{-\Delta_g})(x, y)d\tau + O(\lambda^{-N}).\end{aligned}$$

Let

$$K(t, s) = \frac{1}{\pi}\int \hat{\chi}(\tau)e^{i\lambda\tau}\cos(\tau\sqrt{-\Delta_g})(\gamma(t), \gamma(s))d\tau.$$

Since supp$\hat{\chi} \subset [-1, 1]$ and the injective radius of (M, g) is sufficiently large, we may use the Hadamard parametrix (see [1], [24]) to estimate $K(t, s)$. Let $\kappa(x, y)$ be the vector from x to y in the geodesic normal coordinates at x. Then $d_g(x, y) = |\kappa(x, y)|$. We can write

$$\begin{aligned}\cos(\tau\sqrt{-\Delta_g})(\gamma(t), \gamma(s)) &= (2\pi)^{-3}w(t, s)\int_{\mathbb{R}^3} e^{i\kappa(\gamma(t),\gamma(s))\cdot\xi}\cos(\tau|\xi|)d\xi \\ &+ \sum_{\pm}\int_{\mathbb{R}^3} e^{i\kappa(\gamma(t),\gamma(s))\cdot\xi}e^{i\tau|\xi|}a_\pm(\tau, t, s; |\xi|)d\xi + R(t, s),\end{aligned}$$

where

$$w \in C^\infty[-\tfrac{1}{2}, \tfrac{1}{2}]^2,\ w(t, t) = 1,\ |t| \leqslant \tfrac{1}{2}, \tag{1.2.2}$$

$$|\partial_\tau^j a_\pm(\tau, t, s; |\xi|)| \leqslant C_j(1 + |\xi|)^{-2},\ j = 0, 1, 2, ...,$$

$$|R(t, s)| \leqslant C.$$

As in Chen-Sogge [8], it is not difficult to see that the contributions of the second term and the remainder term to $K(t,s)$ is $O(1)$. Indeed, for each $N \gg 1$ there is a constant C_N such that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\int_{-\infty}^{\infty} \hat{\chi}(\tau)e^{i\kappa(\gamma(t),\gamma(s))\cdot\xi}a_{\pm}(\tau,t,s,|\xi|)e^{i\lambda\tau}e^{\pm i\tau|\xi|}d\tau d\xi\Big| \\ \leqslant C_N\int_{\mathbb{R}^3}(1+|\lambda-|\xi||)^{-N}(1+|\xi|)^{-2}d\xi.\end{aligned}$$

For $N>1$, the last expression is uniformly bounded independent of λ.

Then we have

$$\begin{aligned}K(t,s) &= (2\pi)^{-4}w(t,s)\int_{\mathbb{R}}\int_{\mathbb{R}^3}\hat{\chi}(\tau)e^{i\kappa(\gamma(t),\gamma(s))\cdot\xi}e^{i\tau(\lambda-|\xi|)}d\xi d\tau + O(1) \\ &= (2\pi)^{-3}w(t,s)\int_{\mathbb{R}^3}\chi(\lambda-|\xi|)e^{i\kappa(\gamma(t),\gamma(s))\cdot\xi}d\xi + O(1) \\ &= \frac{w(t,s)}{2\pi^2}\int_0^\infty \chi(\lambda-r)\int_{S^2}e^{ir\kappa(\gamma(t),\gamma(s))\cdot\omega}d\omega r dr + O(1) \\ &= \frac{w(t,s)}{2\pi^2}\int_0^\infty \chi(\lambda-r)\frac{\sin(d_g(\gamma(t),\gamma(s))r)}{d_g(\gamma(t),\gamma(s))}rdr + O(1) \\ &= \frac{w(t,s)}{2\pi^2}\int_0^\infty \chi(\lambda-r)\frac{\sin(d_g(\gamma(t),\gamma(s))r)}{|t-s|}rdr + O(\lambda) \quad (\text{by } \textit{Lemma } 1) \\ &= \frac{1}{2\pi^2}\int_0^\infty \chi(\lambda-r)\frac{\sin(d_g(\gamma(t),\gamma(s))r)}{|t-s|}rdr + O(\lambda) \quad (\textit{by } (1.2.2)).\end{aligned}$$

Let

$$T_\lambda f(t) = \text{p.v.}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{e^{i\lambda\phi(t,s)}}{t-s}f(s)ds,$$

where $\phi(t,s) = d_g(\gamma(t),\gamma(s))\text{sgn}(t-s)$.

Therefore, we will have

$$\|\chi(\lambda-\sqrt{-\Delta_g})\|_{L^2(\gamma)\to L^2(\gamma)} \leqslant C\lambda,$$

if we can show T_λ is uniformly bounded on $L^2[-\frac{1}{2},\frac{1}{2}]$.

1.2.2 End of the proof of the theorems

Now it is clear that Theorem 2 follows immediately from Pan's Proposition 1. So we only give an independent proof of Theorem 1 here.

Let $\beta \in C_0^\infty(\mathbb{R})$ be an even function satisfying

$$\mathrm{supp}\beta \subset [-2, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 2], \ \sum_{j=0}^{\infty} \beta(2^j x) = 1, \ x \in [-1, 1].$$

Let $T_\lambda = \sum_{2^j \leqslant \lambda^{1/3}} T_\lambda^j + R_\lambda$, where

$$T_\lambda^j f(t) = \int \frac{e^{i\lambda\phi(t,s)}}{t-s} \beta(2^j(t-s)) f(s) ds, \ j = 0, 1, 2, ...,$$

$$R_\lambda f(t) = \sum_{2^j \geqslant \lambda^{1/3}} T_\lambda^j f(t) = \text{p.v.} \int \frac{e^{i\lambda\phi(t,s)}}{t-s} \eta(\lambda^{\frac{1}{3}}(t-s)) f(s) ds, \ \eta \in C_0^\infty(\mathbb{R}).$$

First, we show that R_λ is uniformly bounded on $L^2(\mathbb{R})$ by comparing it with a convolution operator. Indeed, the boundedness of the Fourier multiplier

$$\left| \left(e^{i\lambda x} \eta(\lambda^{\frac{1}{3}} x) \text{p.v.} \frac{1}{x} \right)^\wedge (y) \right| \lesssim \|\hat{\eta}\|_1 \approx 1,$$

implies the uniform $L^2 \to L^2$ bound of the convolution operator:

$$f \longmapsto \text{p.v.} \int \frac{e^{i\lambda(t-s)}}{t-s} \eta(\lambda^{\frac{1}{3}}(t-s)) f(s) ds.$$

Then the uniform bound of $\|R_\lambda\|_{L^2 \to L^2}$ follows from

$$\int \frac{|e^{i\lambda\phi(t,s)} - e^{i\lambda(t-s)}|}{|t-s|} |\eta(\lambda^{\frac{1}{3}}(t-s))| ds \lesssim \int_{|t-s| \leqslant \lambda^{1/3}} \lambda |t-s|^2 ds \lesssim 1,$$

and Young's inequality.

Clearly, a trivial bound $\|T_\lambda^j\|_{L^2 \to L^2} \leqslant 1$ is implied by Young's inequality. To show T_λ has a uniform bound, we must refine it for $1 \ll 2^j \leqslant \lambda^{\frac{1}{3}}$. Recall that

$$\phi(t, s) = (t-s) - c(t)(t-s)^3 + d(t, t-s)(t-s)^4.$$

Direct calculation gives

$$\phi_{ts}''(t, s) = 6c(t)(t-s) + O(|t-s|^2),$$

$$\phi'''_{tts}(t,s) = 6c(t) + O(|t-s|),$$

$$\phi''''_{ttts}(t,s) = O(1),$$

if $|t-s| \ll 1$. We may assume that the coefficients in the expansion (1.1.6) satisfy $\|c\|_{C^4} + \|d\|_{C^4} \leqslant 10$. Let

$$\tilde{T}^j_\lambda f(t) = \int e^{i\lambda\phi(t,s)} a(t,s) f(s) ds,$$

where $a(t,s) = (t-s)^{-1}\beta(2^j(t-s))$. The kernel of $\tilde{T}^j_\lambda \tilde{T}^{j*}_\lambda$ is

$$\begin{aligned} K(s,s') &= \int e^{i\lambda(\phi(t,s)-\phi(t,s'))} a(t,s)\overline{a(t,s')} dt \\ &\triangleq \int e^{i\lambda(s-s')\varphi(t,s,s')} \tilde{a}(t,s,s') dt. \end{aligned}$$

Since $|t-s| \approx 2^{-j} \ll 1$, we have $|\phi''_{st}| \approx 2^{-j}$ by our assumptions on c and d. Then by the mean value theorem,

$$|\varphi'_t(t,s,s')| = |\phi''_{st}(t,s'')| \approx 2^{-j}, \tag{1.2.3}$$

where s'' is between s and s'. Moreover, direct calculation shows

$$|\partial^k_t \tilde{a}| \lesssim 2^{(k+2)j}, \ \ |\partial^k_t \phi''_{st}| \lesssim 1, \ \ k = 0,1,2. \tag{1.2.4}$$

Clearly,

$$|K(s,s')| \leqslant \int |a(t,s)||a(t,s')| dt \lesssim 2^j.$$

One the other hand, we can use integration by parts twice to get

$$\begin{aligned} |K(s,s')| &\leqslant (\lambda|s-s'|)^{-2} \int \left| \frac{\partial}{\partial t}\left(\frac{1}{\varphi'_t} \frac{\partial}{\partial t}\left(\frac{\tilde{a}}{\varphi'_t} \right)\right)\right| dt \\ &\lesssim (\lambda|s-s'|)^{-2} 2^{5j}, \end{aligned}$$

where we use the estimates in (1.2.3) and (1.2.4). Then

$$\int |K(s,s')| ds \lesssim \int_0^{2^{-j}} \min\{2^j, (\lambda x)^{-2} 2^{5j}\} dx \lesssim \lambda^{-1} 2^{3j}$$

and Young's inequality give $\|T^j_\lambda\|_{L^2\to L^2} \lesssim \lambda^{-\frac{1}{2}} 2^{\frac{3}{2}j}$ for $1 \ll 2^j \leqslant \lambda^{\frac{1}{3}}$. Then the uniform bound of T_λ follows from a summation.

2

Improved critical eigenfunction restriction estimates on Riemannian manifolds with constant negative curvature

2.1 Introduction

Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geqslant 2$, and let Δ_g be the associated Laplace-Beltrami operator. Let e_λ denote the L^2-normalized eigenfunction

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

so that $\lambda \geqslant 0$ is the eigenvalue of the operator $\sqrt{-\Delta_g}$. A classical result on the L^p-estimates of the eigenfunctions is due to Sogge [23]:

$$\|e_\lambda\|_{L^p(M)} \leqslant C\lambda^{\delta(p)}, \tag{2.1.1}$$

where $2 \leqslant p \leqslant \infty$ and

$$\delta(p) = \begin{cases} \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leqslant p \leqslant p_c, \\ n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & p_c \leqslant p \leqslant \infty, \end{cases}$$

if we set $p_c = \frac{2n+2}{n-1}$. These estimates (2.1.1) are saturated on the round sphere S^n by zonal functions for $p \geqslant p_c$ and for $2 < p \leqslant p_c$ by the highest weight spherical harmonics. However, it is expected that (2.1.1) can be improved for generic Riemannian manifolds. It was known that one can get log improvements for $\|e_\lambda\|_{L^p(M)}$, $p_c < p \leqslant \infty$, when M has nonpositive sectional curvature. Indeed, Bérard's results [1] on improved remainder term bounds for the pointwise Weyl law imply that

$$\|e_\lambda\|_{L^\infty(M)} \leqslant C\lambda^{\frac{n-1}{2}}(\log\lambda)^{-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}.$$

Recently, Hassell and Tacy [9] obtained a similar $(\log\lambda)^{-\frac{1}{2}}$ gain for all $p > p_c$.

Similar L^p-estimates have been established for the restriction of eigenfunctions to geodesic segments. Let $\mathit{\Pi}$ denotes the space of all unit-length geodesics. The works [5], [12], [8](see also [19] for earlier results on hyperbolic surfaces) showed that

$$\sup_{\gamma\in\mathit{\Pi}}\Big(\int_\gamma |e_\lambda|^p\,ds\Big)^{\frac{1}{p}} \leqslant C\lambda^{\sigma(n,p)}\|e_\lambda\|_{L^2(M)}, \tag{2.1.2}$$

where

$$\sigma(2,p) = \begin{cases} \frac{1}{4}, & 2 \leqslant p \leqslant 4, \\ \frac{1}{2} - \frac{1}{p}, & 4 \leqslant p \leqslant \infty, \end{cases} \tag{2.1.3}$$

$$\sigma(n,p) = \frac{n-1}{2} - \frac{1}{p}, \text{ if } p \geqslant 2 \text{ and } n \geqslant 3. \tag{2.1.4}$$

It was known that these estimates are saturated by the highest weight spherical harmonics when $n \geqslant 3$ on round sphere S^n, as well as in the case of $2 \leqslant p \leqslant 4$ when $n = 2$, while in this case the zonal functions saturate the bounds for $p \geqslant 4$.

There are considerable works towards improving (2.1.2) for the 2-dimensional manifolds with nonpositive curvature. Chen [7] proved a $(\log\lambda)^{-\frac{1}{2}}$ gain for all $p > 4$. Sogge and Zelditch [25] and Chen and Sogge [8] showed that one can improve (2.1.2) for $2 \leqslant p \leqslant 4$, in the sense that

$$\sup_{\gamma\in\mathit{\Pi}}\Big(\int_\gamma |e_\lambda|^p\,ds\Big)^{\frac{1}{p}} = o(\lambda^{\frac{1}{4}}). \tag{2.1.5}$$

Recently, using the Toponogov's comparison theorem, Blair and Sogge [3] obtained log improvements for $p=2$:

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^2\,ds\Big)^{\frac12} \leqslant C\lambda^{\frac14}(\log\lambda)^{-\frac14}\|e_\lambda\|_{L^2(M)}. \tag{2.1.6}$$

Inspired by the works [8], [3], [22], Xi and the author [27] was able to deal with the other endpoint $p=4$ and proved a $(\log\log\lambda)^{-\frac18}$ gain for surfaces with nonpositive curvature and a $(\log\lambda)^{-\frac14}$ gain for hyperbolic surfaces

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^4\,ds\Big)^{\frac14} \leqslant C\lambda^{\frac14}(\log\lambda)^{-\frac14}\|e_\lambda\|_{L^2(M)}. \tag{2.1.7}$$

In the 3-dimensional case, under the assumption of nonpositive curvature, Chen [7] also proved a $(\log\lambda)^{-\frac12}$ gain for all $p>2$. With the assumption of constant negative curvature, Chen and Sogge [8] showed that

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^2\,ds\Big)^{\frac12} = o(\lambda^{\frac12}). \tag{2.1.8}$$

Moreover, Hezari and Rivière [11] and Hezari [10] used quantum ergodic methods to get logarithmic improvements at critical exponents in the cases above on negatively curved manifolds for a density one subsequence.

The purpose of this chapter is to prove a $(\log\lambda)^{-\frac12}$ gain for the L^2 geodesic restriction bounds on 3-dimensional compact Riemannian manifolds with constant negative curvature. We mainly follow the approaches developed in [8], [3], [27]. We derive an explicit formula for the wave kernel on $\mathbb{H}^3$, which is one of the key steps to get the $(\log\lambda)^{-\frac12}$ gain. We shall lift all the calculations to the universal cover $\mathbb{H}^3$ and then use the Poincaré half-space model to derive the explicit formulas of the mixed derivatives of the distance function restricted to the unit geodesic segments. Then we decompose the domain of the distance function and compute the bounds of various mixed derivatives explicitly, since it was observed in [8] and [27] that the desired kernel estimates follow from the oscillatory integral estimates and the estimates on the mixed derivatives. Moreover, whether one can get similar logarithmic improvements on 3-dimensional manifolds with nonpositive curvature is still an interesting open problem. One of the technical difficulties is that these manifolds may not have sufficiently many totally geodesic submanifolds (see [8, p.458]). Throughout this chapter, we shall assume that the injectivity radius of M is sufficiently large, and fix γ to be a unit length geodesic segment parameterized by arclength.

Theorem 3. *Let (M,g) be a 3-dimensional compact Riemannian manifold of constant negative curvature, let $\gamma\subset M$ be a fixed unit-length geodesic segment. Then for $\lambda\gg1$, there is a constant C*

such that

$$\|e_\lambda\|_{L^2(\gamma)} \leqslant C\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}. \tag{2.1.9}$$

Moreover, if Π denotes the set of unit-length geodesics, there exists a uniform constant $C = C(M,g)$ such that

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^2\,ds\Big)^{\frac{1}{2}} \leqslant C\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}. \tag{2.1.10}$$

Remark 2. As a final remark, we must mention a recently posted work of Blair [2]. He was able to use geometric tools different from ours to establish bounds on the mixed partials of the distance function on the covering manifold restricted to geodesic segements. Then he independently proved (2.1.7) for surfaces with general nonpositive curvature and a $(\log\lambda)^{-\frac{1}{2}+\epsilon}$ gain for (2.1.8) on 3-dimensional manifolds with constant negative curvature. Moreover, recently Professor C. Sogge pointed out to the author that one may also get a similar $(\log\lambda)^{-\frac{1}{2}}$ gain for the L^4 geodesic restriction estimates on surfaces with strictly negative curvatures by using the Günther's comparison theorem and the Hadamard parametrix.

2.2 Preliminaries

We start with some standard reductions. Since the uniform bound (2.1.10) follows from a standard compactness argument in [8, p.452], we only need to prove (2.1.9). Let $T \gg 1$. Let $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = 1$ and supp $\hat{\rho} \subset [-1/2, 1/2]$, then it is clear that the operator $\rho(T(\lambda - \sqrt{-\Delta_g}))$ reproduces eigenfunctions, namely

$$\rho(T(\lambda - \sqrt{-\Delta_g}))e_\lambda = e_\lambda.$$

Let $\chi = |\rho|^2$. After a standard TT^* argument, we only need to estimate the norm

$$\|\chi(T(\lambda - \sqrt{-\Delta_g}))\|_{L^2(\gamma)\to L^2(\gamma)}. \tag{2.2.1}$$

Choose a bump function $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau) = 1 \quad \text{for} \quad |\tau| \leqslant 3/2, \quad \text{and} \quad \beta(\tau) = 0, \quad |\tau| \geqslant 2.$$

By the Fourier inversion formula, we may represent the kernel of the operator $\chi(T(\lambda - \sqrt{-\Delta_g}))$ as an operator valued integral

$$\begin{aligned}\chi(T(\lambda - \sqrt{-\Delta_g}))(x, y) &= \frac{1}{2\pi T}\int \beta(\tau)\hat{\chi}(\tau/T)e^{i\lambda\tau}(e^{-i\tau\sqrt{-\Delta_g}})(x, y)d\tau \\ &+\frac{1}{2\pi T}\int (1 - \beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(e^{-i\tau\sqrt{-\Delta_g}})(x, y)d\tau = K_0(x, y) + K_1(x, y).\end{aligned}$$

Then one may use a parametrix to estimate the norm of the integral operator associated with the kernel $K_0(\gamma(t), \gamma(s))$ (see [8, p.455])

$$\|K_0\|_{L^2[0,1]\to L^2[0,1]} \leqslant C\lambda T^{-1}. \tag{2.2.2}$$

Since the kernel of $\chi(T(\lambda+\sqrt{-\Delta_g}))$ is $O(\lambda^{-N})$ with constants independent of T, by Euler's formula we are left to consider the integral operator S_λ:

$$S_\lambda h(t) = \frac{1}{\pi T}\int_{-\infty}^{\infty}\int_0^1 (1 - \beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(\cos\tau\sqrt{-\Delta_g})(\gamma(t), \gamma(s))h(s)dsd\tau. \tag{2.2.3}$$

As in [8], [3], [27], we use the Hadamard parametrix and the Cartan-Hadamard theorem to lift the calculations up to the universal cover $(\mathbb{R}^3, \tilde{g})$ of (M, g). Let Γ denote the group of deck transformations preserving the associated covering map $\kappa : \mathbb{R}^3 \to M$ coming from the exponential map from $\gamma(0)$ associated with the metric g on M. The metric $\tilde{g}$ is its pullback via κ. Choose also a Dirchlet fundamental domain, $D \simeq M$, for M centered at the lift $\tilde{\gamma}(0)$ of $\gamma(0)$. Let $\tilde{\gamma}(t)$, $t \in \mathbb{R}$, satisfy $\kappa(\tilde{\gamma}(t)) = \gamma(t)$, where γ is the unit speed geodesic containing the geodesic segment $\{\gamma(t) : \ t \in [0,1]\}$. Then $\tilde{\gamma}(t)$ is also a geodesic parameterized by arclength. We measure the distances in $(\mathbb{R}^3, \tilde{g})$ using its Riemannian distance function $d_{\tilde{g}}(\,\cdot\, , \,\cdot\,)$. Moreover, we recall that if $\tilde{x}$ denotes the lift of $x \in M$ to D, then

$$(\cos t\sqrt{-\Delta_g})(x,y) = \sum_{\alpha \in \Gamma} (\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})).$$

Hence for $t \in [0,1]$,

$$S_\lambda h(t) = \frac{1}{\pi T} \sum_{\alpha \in \Gamma} \int_{\mathbb{R}} \int_0^1 (1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(\cos \tau\sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))h(s)\, dsd\tau.$$

As in [3] and [27], we denote the R-tube about the infinite geodesic $\tilde{\gamma}$ by

$$\mathrm{T}_R(\tilde{\gamma}) = \{(x,y,z) \in \mathbb{R}^3 : d_{\tilde{g}}((x,y,z),\tilde{\gamma}) \leqslant R\} \tag{2.2.4}$$

and

$$\Gamma_{\mathrm{T}_R(\tilde{\gamma})} = \{\alpha \in \Gamma : \alpha(D) \cap \mathrm{T}_R(\tilde{\gamma}) \neq \varnothing\}.$$

From now on we fix $R \approx \mathrm{Inj}M$. We will see that $\mathrm{T}_R(\tilde{\gamma})$ plays a key role in the proof of Lemma 4. Then we decompose the sum

$$S_\lambda h(t) = S_\lambda^{tube} h(t) + S_\lambda^{osc} h(t) = \sum_{\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}} S_{\lambda,\alpha}^{tube} h(t) + \sum_{\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}} S_{\lambda,\alpha}^{osc} h(t), t \in [0,1].$$

Then by the finite propagation speed property and $\hat{\chi}(\tau) = 0$ if $|\tau| \geqslant 1$, we have

$$d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \leqslant T, s, t \in [0,1].$$

As observed in [3, p.11],

$$\#\{\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})} : d_{\tilde{g}}(0, \alpha(0)) \in [2^k, 2^{k+1}]\} \leqslant C2^k. \tag{2.2.5}$$

Thus the number of nonzero summands in $S_\lambda^{tube} h(t)$ is $O(T)$ and in $S_\lambda^{osc} h(t)$ is $O(e^{CT})$.

Given $\alpha \in \Gamma$ set with $s, t \in [0,1]$

$$K_\alpha(t,s) = \frac{1}{\pi T}\int_{-T}^{T} (1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(\cos\tau\sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))d\tau.$$

When $\alpha = Identity$, one can use the Hadamard parametrix to prove the same bound as (2.2.2) (see e.g. [7], p. 9)

$$\|K_{\mathrm{Id}}\|_{L^2[0,1]\to L^2[0,1]} \leqslant C\lambda T^{-1}. \tag{2.2.6}$$

If $\alpha \neq Identity$, we set $\phi(t,s) = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))$, $s, t \in [0,1]$. Then by finite propagation speed and $\alpha \neq Identity$, we have

$$2 \leqslant \phi(t,s) \leqslant T, \quad \text{if} \quad s,t \in [0,1]. \tag{2.2.7}$$

As in [8, p.456], one may use the Hadamard parametrix and stationary phase to show that $|K_\alpha(t,s)| \leqslant C\lambda T^{-1}r^{-1} + e^{CT}$, where $r = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))$. However, we may get a much better estimate for K_α. To see this, we need to derive the explicit formula of the wave kernel on hyperbolic space. Without loss of generality, we may assume that (M,g) has constant negative curvature -1, which implies that the covering manifold $(\mathbb{R}^3, \tilde{g})$ is the hyperbolic space $\mathbb{H}^3$. If we denote the shifted Laplacian operator by

$$L = \Delta_{\tilde{g}} + \frac{(n-1)^2}{4} = \Delta_{\tilde{g}} + 1 \qquad (\text{for } n = 3),$$

which has the property $\mathrm{Spec}(-L) = [0, \infty)$, then there are exact formulas for various functions of L (see e.g. [26, Chapter 8, (5.15)]). Indeed,

$$h(\sqrt{-L})\delta_y(x) = -\frac{1}{(2\pi)^{3/2}}\frac{1}{\sinh r}\frac{\partial}{\partial r}\hat{h}(r),$$

where $\hat{h}$ is the Fourier transform defined by

$$\hat{h}(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(k)e^{-irk}dk.$$

If $h(k) = \frac{\sin(tk)}{k}$, then $\hat{h}(r) = \frac{\sqrt{2\pi}}{2}\mathbf{1}_{\{r\leqslant|t|\}}$. Hence, for $t > 0$,

$$\frac{\sin t\sqrt{-L}}{\sqrt{-L}}\delta_y(x) = \frac{\delta(t-r)}{4\pi\sinh r}, \tag{2.2.8}$$

where $x,\ y \in \mathbb{H}^3$ and $r = d_{\tilde{g}}(x,y)$. Differentiating it yields

$$\cos t\sqrt{-L}\ \delta_y(x) = \frac{\delta'(t-r)}{4\pi \sinh r}. \tag{2.2.9}$$

Recall the following relation between L and $\Delta_{\tilde{g}}$ (see e.g. [13, Proposition 2.1])

$$\cos t\sqrt{-\Delta_{\tilde{g}}} = \cos t\sqrt{-L} - t\int_0^t \frac{J_1(\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}} \cos s\sqrt{-L} ds, \tag{2.2.10}$$

where $J_1(v)$ is the Bessel function

$$J_1(v) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\Big(\frac{v}{2}\Big)^{2k+1}.$$

We plug (2.2.9) into the relation (2.2.10) to see that for $t > 0$,

$$\cos t\sqrt{-\Delta_{\tilde{g}}}\ \delta_y(x) = \frac{\delta'(t-r)}{4\pi \sinh r} - t\int_0^t \frac{J_1(\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}} \frac{\partial_s \delta(s-r)}{4\pi \sinh r} ds,$$

Thus, integrating by parts and noting that $\cos t\sqrt{-\Delta_{\tilde{g}}}$ is even in t, we get the following explicit formula for the wave kernel "$\cos t\sqrt{-\Delta_{\tilde{g}}}(x,y)$" on $\mathbb{H}^3$

$$\cos t\sqrt{-\Delta_{\tilde{g}}}\ \delta_y(x) = \frac{1}{4\pi \sinh r}\Big[\delta'(|t|-r) - J_1'(0)|t|\delta(|t|-r) - \frac{r|t|G'(\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\mathbf{1}_{\{r\leqslant |t|\}}\Big], \tag{2.2.11}$$

where $t \in \mathbb{R}\backslash\{0\}$, and $G(v) = J_1(v)/v$ is an entire function of v^2, satisfying

$$G(v) \sim Cv^{-3/2}\cos\Big(v - \frac{3\pi}{4}\Big) + \cdots, \text{ as } v \to +\infty. \tag{2.2.12}$$

Lemma 2. *If $\alpha \neq$ Identity, we have*

$$|K_\alpha(t,s)| \leqslant C\lambda T^{-1}e^{-r/2}, \text{ for } t,s \in [0,1],$$

where $r = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \geqslant 1$ and C is a constant independent of T and r.

Using this lemma and (2.2.5), we get

$$\Big|\sum_{\alpha\in\Gamma_{\mathrm{T}_R(\tilde{\gamma})}\backslash\{\mathrm{Id}\}} K_\alpha(t,s)\Big| \leqslant C\lambda T^{-1}\sum_{1\leqslant 2^k\leqslant T} 2^k e^{-2^k/2} \leqslant C\lambda T^{-1}. \tag{2.2.13}$$

Consequently, by Young's inequality and the estimate on K_{Id} (2.2.6) we have

$$\|S_\lambda^{tube}\|_{L^2[0,1]\to L^2[0,1]} \leqslant C\lambda T^{-1}. \tag{2.2.14}$$

Proof of Lemma 2. Since the formula of the wave kernel (2.2.11) consists of 3 terms, we should estimate their contributions separately. Integrating by parts yields

$$\begin{aligned}\Big|\int_{-T}^{T}(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}\delta'(|\tau|-r)d\tau\Big| &\leqslant \sum_{\tau=\pm r}\Big|\frac{d}{d\tau}\Big[(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}\Big]\Big| \\ &\leqslant C\lambda,\end{aligned} \tag{2.2.15}$$

since $\beta,\ \hat{\chi}\in\mathcal{S}(\mathbb{R})$. Similarly,

$$\begin{aligned}\Big|\int_{-T}^{T}(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}|\tau|\delta(|\tau|-r)d\tau\Big| &= \Big|\sum_{\tau=\pm r}(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}|\tau|\Big| \\ &\leqslant Cr.\end{aligned} \tag{2.2.16}$$

Noting that $J_1(v)$, $J_1'(v)$ are uniformly bounded for $v\in\mathbb{R}$ and $G(v)$ is an entire function of v^2, we see that $G'(v)/v$ is also uniformly bounded for $v\in\mathbb{R}$. Moreover, by (2.2.12), there is some $N\gg 1$ such that

$$|G'(v)/v|\leqslant Cv^{-5/2},\quad \text{for } v>N.$$

This gives

$$\begin{aligned}&\Big|\int_{-T}^{T}(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}\frac{r|\tau|G'(\sqrt{\tau^2-r^2})}{\sqrt{\tau^2-r^2}}\mathbf{1}_{\{r\leqslant|\tau|\}}d\tau\Big| \\ &\leqslant Cr\int_{|\tau|\geqslant r}|\tau|\,\Big|\frac{G'(\sqrt{\tau^2-r^2})}{\sqrt{\tau^2-r^2}}\Big|\,d\tau \\ &\leqslant Cr\Big(\int_0^N|\rho+r|d\rho+\int_N^\infty|\rho+r||\rho|^{-5/2}d\rho\Big) \\ &\leqslant Cr(C+Cr),\end{aligned} \tag{2.2.17}$$

where $\rho=|\tau|-r$. Hence

$$|K_\alpha(t,s)|\leqslant\frac{C\lambda+Cr+Cr^2}{T\sinh r}\leqslant C\lambda T^{-1}e^{-r/2}.$$

$\square$

Remark 3. As pointed out in Remark 2, one may also obtain Lemma 2 by using the Hadamard parametrix and the Günther's comparison theorem.

2.3 Proof of the main theorem

2.3.1 Oscillatory integral theorems

Now we are left to estimate the kernels $K_\alpha(t,s)$ with $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. From now on, we assume that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. First of all, we need a slight variation of the oscillatory integral theorem in [27, Proposition 2]. Indeed, it is a detailed version of the estimates by Phong and Stein [18] on the oscillatory integrals with fold singularities.

Proposition 2. *Let $a \in C_0^\infty(\mathbb{R}^2)$, let $\phi \in C^\infty(\mathbb{R}^2)$ be real valued and $\lambda > 0$, set*

$$T_\lambda f(t) = \int_{-\infty}^{\infty} e^{i\lambda\phi(t,s)} a(t,s) f(s) ds, \ \ f \in C_0^\infty(\mathbb{R}).$$

If $\phi''_{st} \neq 0$ on $\operatorname{supp} a$, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leqslant C_{a,\phi} \lambda^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C_{a,\phi} = C\operatorname{diam}(\operatorname{supp} a)^{\frac{1}{2}} \left\{ \|a\|_\infty + \frac{\sum_{0\leqslant i,j\leqslant 2} \|\partial_t^i a\|_\infty \|\partial_t^j \phi''_{st}\|_\infty}{\inf |\phi''_{st}|^2} \right\}. \tag{2.3.1}$$

Assume $\operatorname{supp} a$ is contained in some compact set $F \subseteq \mathbb{R}^2$. Denote the ranges of t and s in F by $F_t \subseteq \mathbb{R}$ and $F_s \subseteq \mathbb{R}$ respectively. If for any $s \in F_s$, there is a unique $t_c = t_c(s) \in F_t$ such that $\phi''_{st}(t_c, s) = 0$, and if $\phi'''_{stt}(t_c, s) \neq 0$ on F_s, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leqslant C'_{a,\phi} \lambda^{-\frac{1}{4}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C'_{a,\phi} = C\operatorname{diam}(\operatorname{supp} a)^{\frac{1}{4}} \left\{ \|a\|_\infty + \frac{\sum_{0\leqslant i,j\leqslant 2} \|\partial_t^i a\|_\infty \|\partial_t^j \phi''_{st}\|_\infty}{\inf |\phi''_{st}/(t-t_c(s))|^2} \right\}. \tag{2.3.2}$$

Dually, if for any $t \in F_t$, there is a unique $s_c = s_c(t) \in F_s$ such that $\phi''_{st}(t, s_c) = 0$, and if $\phi'''_{tss}(t, s_c) \neq 0$ on F_t, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leqslant C''_{a,\phi} \lambda^{-\frac{1}{4}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C''_{a,\phi} = C\operatorname{diam}(\operatorname{supp} a)^{\frac{1}{4}} \left\{ \|a\|_\infty + \frac{\sum_{0\leqslant i,j\leqslant 2} \|\partial_s^i a\|_\infty \|\partial_s^j \phi''_{st}\|_\infty}{\inf |\phi''_{st}/(s-s_c(t))|^2} \right\}. \tag{2.3.3}$$

The L^∞-norm and the infimum are taken on $\operatorname{supp} a$. The constant $C > 0$ is independent of λ, a, ϕ

and F.

Proof. Noting that the first part is due to non-stationary phase (see [27, p.15]) and the third part simply follows from duality, we only need to prove the second part. As in [27, p.15], by a TT^* argument, it suffices to estimate the kernel of $T_\lambda^* T_\lambda$

$$K(s,s') = \int e^{i\lambda(\phi(t,s)-\phi(t,s'))} a(t,s)\overline{a(t,s')}dt.$$

Let

$$\varphi(t,s,s') = \frac{\phi(t,s)-\phi(t,s')}{s-s'}, \text{ for } s \neq s', \text{ and } \varphi(t,s,s) = \phi'_s(t,s),$$

$$\tilde{a}(t,s,s') = a(t,s)\overline{a(t,s')}.$$

Then the kernel has the form

$$K(s,s') = \int e^{i\lambda(s-s')\varphi(t,s,s')}\tilde{a}(t,s,s')dt. \tag{2.3.4}$$

Using the mean value theorem, we have $\varphi'_t(t,s,s') = \phi''_{st}(t,s'')$, where s'' is a number between s and s'. By our assumptions, we see that there is a unique point $t_c(s'') \in F_t$ such that $\phi''_{st}(t_c(s''),s'') = 0$, and $\phi'''_{stt}(t_c(s''),s'') \neq 0$. Let $\theta > 0$. Select $\eta \in C_0^\infty(\mathbb{R})$ satisfying $\eta(t) = 1, |t| \leqslant 1$, and $\eta(t) = 0, |t| \geqslant 2$. Then we decompose the oscillatory integral into two parts. First,

$$\Big|\int e^{i\lambda(s-s')\varphi}\tilde{a}\eta((t-t_c(s''))/\theta)dt\Big| \leqslant 4\theta\|a\|_\infty^2.$$

Then integrating by parts yields if $s \neq s'$,

$$\begin{aligned}
&\Big|\int e^{i\lambda(s-s')\varphi}\tilde{a}(1-\eta((t-t_c(s''))/\theta))dt\Big| \\
&\leqslant (\lambda|s-s'|)^{-2} \int\limits_{|t-t_c(s'')|>\theta} \left|\frac{\partial}{\partial t}\left(\frac{1}{\varphi'_t}\frac{\partial}{\partial t}\left(\frac{\tilde{a}(1-\eta((t-t_c(s''))/\theta))}{\varphi'_t}\right)\right)\right| dt \\
&\leqslant \frac{C\Big(\sum\limits_{0\leqslant i,j\leqslant 2}||\partial_t^i a||_\infty||\partial_t^i\phi''_{st}||_\infty\Big)^2}{(\lambda|s-s'|)^2\cdot \inf\left(|\phi''_{st}|/|t-t_c(s)|\right)^4} \int\limits_{|t-t_c(s'')|>\theta} (|t-t_c(s'')|^{-4}+\theta^{-2}|t-t_c(s'')|^{-2})dt \\
&\leqslant C\theta^{-3}(\lambda|s-s'|)^{-2}\frac{\Big(\sum\limits_{0\leqslant i,j\leqslant 2}||\partial_t^i a||_\infty||\partial_t^i\phi''_{st}||_\infty\Big)^2}{\inf\left(|\phi''_{st}|/|t-t_c(s)|\right)^4},
\end{aligned}$$

where C is a constant independent of λ, a, ϕ and F. If we set $\theta = (\lambda|s-s'|)^{-\frac{1}{2}}$, then

$$|K(s,s')| \leqslant C\left\{\|a\|_\infty^2 + \frac{\left(\sum_{0\leqslant i,j\leqslant 2} ||\partial_t^i a||_\infty ||\partial_t^j \phi_{st}''||_\infty\right)^2}{\inf(|\phi_{st}''|/|t-t_c(s)|)^4}\right\}(\lambda|s-s'|)^{-\frac{1}{2}}, \quad \text{if } s \neq s'.$$

Hence,

$$\int |K(s,s')|ds \leqslant C_{a,\phi}'^2 \lambda^{-\frac{1}{2}},$$

which completes the proof by Young's inequality. □

From now on, we will use C to denote various positive constants independent of T. Using the Hadamard parametrix and stationary phase [8, p.446], we can write

$$K_\alpha(t,s) = w(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \sum_{\pm} a_\pm(T,\lambda;\phi(t,s))e^{\pm i\lambda\phi(t,s)} + R_\alpha(t,s),$$

where $|w(x,y)| \leqslant C$, and for each $j = 0,1,2,...$, there is a constant C_j independent of T, $\lambda \geqslant 1$ so that

$$\left|\partial_r^j a_\pm(T,\lambda;r)\right| \leqslant C_j T^{-1}\lambda r^{-1-j}, \;\; r \geqslant 1. \tag{2.3.5}$$

From the Hadamard parametix with an estimate on the remainder term (see [24]), we see that with a uniform constant C

$$|R_\alpha(t,s)| \leqslant e^{CT}.$$

Noting that diam(supp $a_\pm$) $\leqslant 2$ and we have good control on the size of $a_\pm$ and its derivatives by (2.3.5), it remains to estimate the size of ϕ_{st}'' and its derivatives. Without loss of generality, we may assume that (M,g) is a compact 3-dimensional Riemannian manifold with constant curvature equal to -1. As in [27], we will compute the various mixed derivatives of the distance function explicitly on its universal cover $\mathbb{H}^3$.

2.3.2 The Poincaré half space model

We consider the Poincaré half-space model

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\},$$

with the metric $ds^2 = z^{-2}(dx^2 + dy^2 + dz^2)$. Recall that the distance function for the Poincaré half-space model is given by

$$\operatorname{dist}((x_1, y_1, z_1), (x_2, y_2, z_2)) = \operatorname{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{2z_1 z_2}\right),$$

where arcosh is the inverse hyperbolic cosine function

$$\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1}),\ x \geqslant 1.$$

Moreover, the geodesics are the straight vertical rays normal to the $z = 0$-plane and the half-circles normal to the $z = 0$-plane with origins on the $z = 0$-plane. Without loss of generality, we may assume that $\tilde{\gamma}$ is the z-axis. Let $\tilde{\gamma}(t) = (0, 0, e^t),\ t \in \mathbb{R}$, be the infinite geodesic parameterized by arclength. Our unit geodesic segment is given by $\tilde{\gamma}(t),\ t \in [0, 1]$. Then its image $\alpha(\tilde{\gamma}(s)),\ s \in [0, 1]$, is a unit geodesic segment of $\alpha(\tilde{\gamma})$. As before, we denote the distance function $d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))$ by $\phi(t, s)$. Since we are assuming $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$, we have

$$2 \leqslant \phi(t, s) \leqslant T, \quad \text{if} \quad s,\ t \in [0, 1]. \tag{2.3.6}$$

If $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are contained in a common plane, it is reduced to the 2-dimensional case. We recall the following lemma from [27, Lemma 5, 6], where $\tilde{\gamma}(t) = (0, e^t)$ in the Poincarè half-plane model.

Lemma 3. *Let $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. If $\alpha(\tilde{\gamma}) \cap \tilde{\gamma} = \varnothing$, we have*

$$\inf |\phi''_{st}| \geqslant e^{-CT}.$$

Assume that $\alpha(\tilde{\gamma})$ is a half-circle intersecting $\tilde{\gamma}$ at the point $(0, e^{t_0})$, $t_0 \in \mathbb{R}$. If $t_0 \notin [-1, 2]$, which means the intersection point $(0, e^{t_0})$ is outside some neighbourhood of the geodesic segment $\{\tilde{\gamma}(t) : t \in [0, 1]\}$, then we also have

$$\inf |\phi''_{st}| \geqslant e^{-CT}.$$

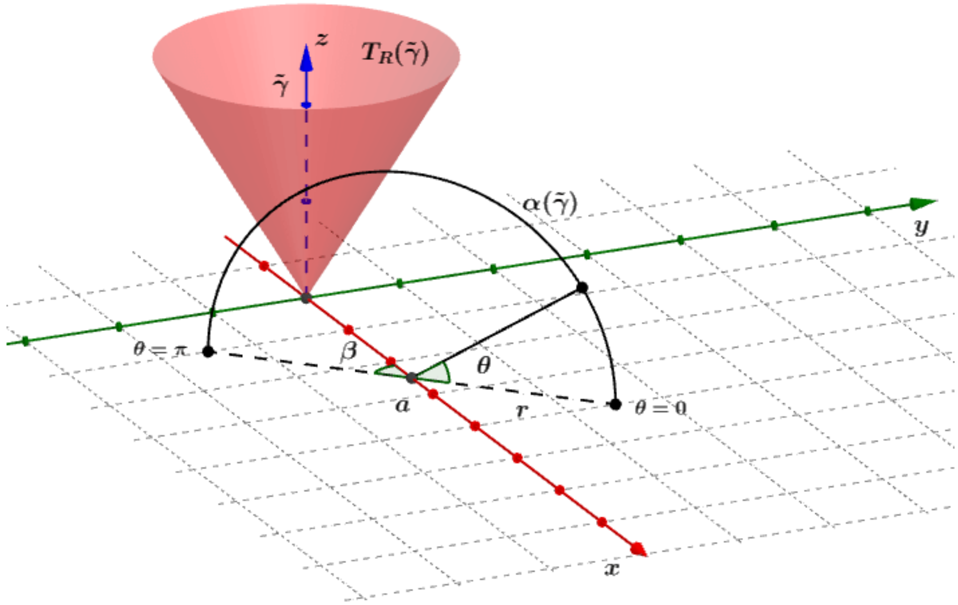

Figure 2.1: Poincaré half-space model

If $t_0 \in [-1, 2]$, *then*

$$\inf |\phi''_{st}/(t - t_0)| \geqslant e^{-CT}.$$

Moreover,

$$\|\phi''_{st}\|_\infty + \|\phi'''_{stt}\|_\infty + \|\phi''''_{sttt}\|_\infty \leqslant e^{CT},$$

where $C > 0$ *is independent of* T. *The infimum and the norm are taken on the unit square* $\{(t, s) \in \mathbb{R}^2 : t, s \in [0, 1]\}$.

From now on, we assume that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$, and $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are not contained in a common plane. Without loss of generality, we set $a \geqslant 0$, $r > 0$, and $\beta \in (0, \frac{\pi}{2}]$. Indeed, one can properly choose a coordinate system to achieve this. Let $\gamma_1(t) = (0, 0, e^t)$, and $\gamma_2(s) = (a + \frac{1-e^{2s}}{1+e^{2s}} r\cos\beta, \frac{1-e^{2s}}{1+e^{2s}} r\sin\beta, \frac{2re^s}{1+e^{2s}})$. It is not difficult to verify that both of them are parameterized by arclength. Assume that

$$\{\tilde{\gamma}(t) : t \in [0, 1]\} = \{\gamma_1(t) : t \in [0, 1]\}, \quad \{\alpha(\tilde{\gamma}(s)) : s \in [0, 1]\} = \{\gamma_2(s) : s \in I\},$$

where I is some unit closed interval of $\mathbb{R}$. Here $\gamma_2(s)$, $s \in \mathbb{R}$, is a half circle centered at $(a, 0, 0)$ with radius r. β is the angle between the y-axis and the normal vector of the plane containing the half circle. Moreover, these two geodesics are contained in a common plane when $\beta = 0$. See Figure 2.1.

Now we are ready to compute $\phi_{st}^{''}$ explicitly and analyze its zero set. For simplification, we denote

$$d_1 = \sqrt{a^2 + r^2 - 2ar\cos\beta} \quad \text{and} \quad d_2 = \sqrt{a^2 + r^2 + 2ar\cos\beta}.$$

Direct computation gives

$$\phi(t,s) = d_{\tilde{g}}(\gamma_1(t), \gamma_2(s)) = \operatorname{arccosh}\Big(\frac{A}{4re^{s+t}}\Big), \; t \in [0,1], \; s \in I,$$

where $A = e^{2s+2t} + e^{2t} + d_1^2 e^{2s} + d_2^2$. Taking derivatives yields

$$\phi_{st}^{''} = \frac{16re^{2s+2t}[(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})]}{(A^2 - 16r^2 e^{2s+2t})^{3/2}}. \tag{2.3.7}$$

The computation is technical. To see (2.3.7), we write

$$e^{s+t}\cosh\phi = \frac{A}{4r}.$$

Taking derivatives on both sides, we obtain

$$(\phi_t' + \phi_s' + \phi_{ts}^{''})\sinh\phi + (1 + \phi_t'\phi_s')\cosh\phi = e^{s+t}/r. \tag{2.3.8}$$

Denote $P = e^{s+t}$, $Q = d_1^2 e^{s-t}$, $R = e^{t-s}$, and $S = d_2^2 e^{-s-t}$. Since

$$4r\cosh\phi = P + Q + R + S,$$

taking derivatives yields

$$4r\phi_t'\sinh\phi = P - Q + R - S, \quad 4r\phi_s'\sinh\phi = P + Q - R - S.$$

Then we multiply both sides of (2.3.8) by $4r^2(\sinh\phi)^2$ and use the hyperbolic trigonometric identity $(\sinh\phi)^2 = (\cosh\phi)^2 - 1$ to obtain

$$4r^2(\sinh\phi)^3\phi_{st}^{''} = (a\cos\beta - r)(P + S) + (a\cos\beta + r)(Q + R).$$

This gives our desired expression (2.3.7).

We denote the zero set of $\phi_{st}^{''}$ by Z. Clearly, if $r \leqslant a\cos\beta$, then $Z = \varnothing$. Assume that $r > a\cos\beta$.

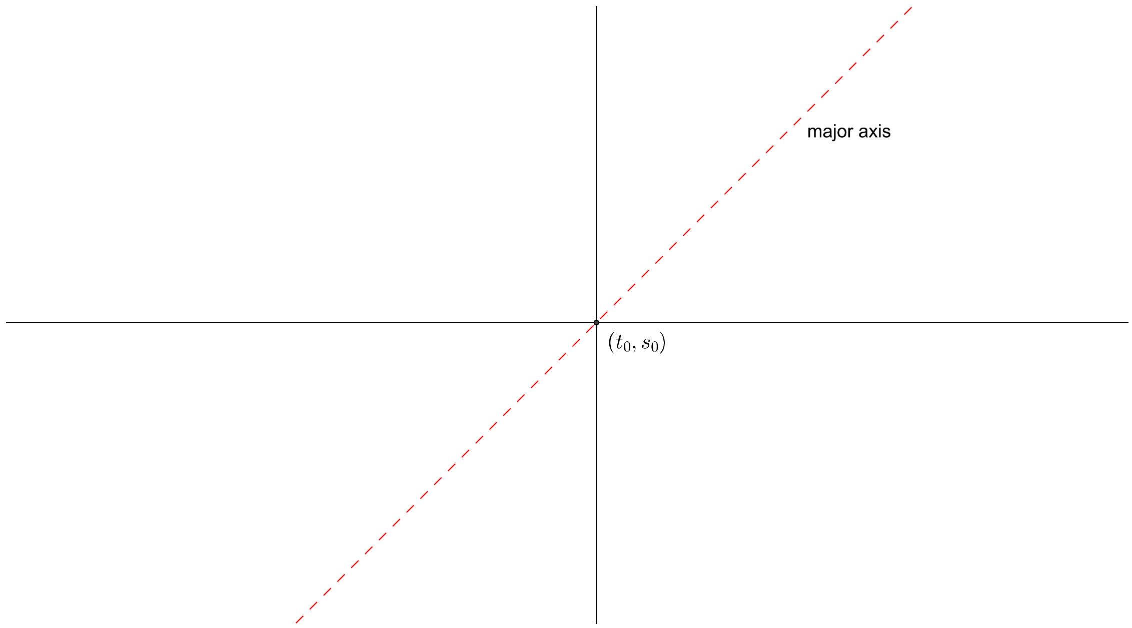

Figure 2.2: Zero set of ϕ''_{st}, $\beta = \frac{\pi}{2}$

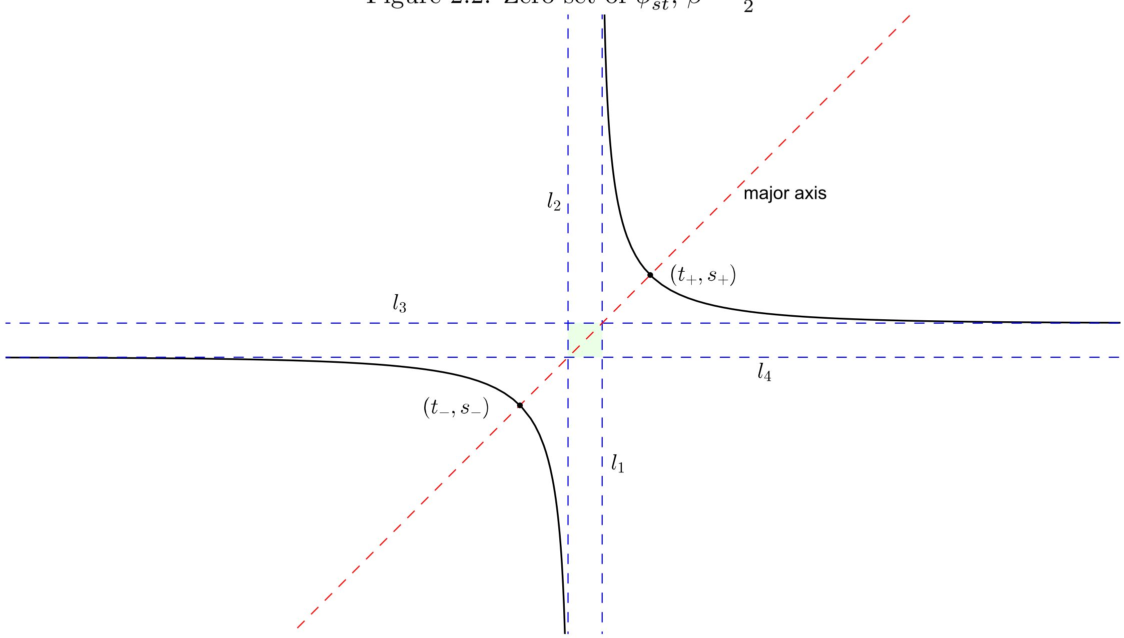

Figure 2.3: Zero set of ϕ''_{st}, $\beta \in (0, \frac{\pi}{2})$

In the interesting special case $\beta = \frac{\pi}{2}$,

$$Z = \{(t, s) \in \mathbb{R}^2 : t = t_0 \text{ or } s = s_0\},$$

where $e^{2t_0} = a^2 + r^2$ and $e^{2s_0} = 1$. See Figure 2.2. In this case, we can easily see that ϕ'''_{stt} and ϕ'''_{tss} vanish at the point (t_0, s_0), as observed in [8, p.454]. In general, if $0 < \beta \leqslant \frac{\pi}{2}$, we have

$$Z = \{(t, s) \in \mathbb{R}^2 : (e^{2t} - X_0)(e^{2s} - Y_0) = B\}, \tag{2.3.9}$$

where

$$Y_0 = \frac{r + a\cos\beta}{r - a\cos\beta}, \ X_0 = d_1^2 Y_0, \ B = \frac{4a^3 r\cos\beta\sin^2\beta}{(r - a\cos\beta)^2}, \tag{2.3.10}$$

and

$$X_0 Y_0 - B = d_2^2. \tag{2.3.11}$$

When $\beta \in (0, \frac{\pi}{2})$, the set Z consists of two disconnected curves. See Figure 2.3. It has four different asymptotes:

$$l_1 : \ t = \ln\sqrt{X_0}, \quad l_2 : \ t = \ln\sqrt{X_0 - B/Y_0},$$

$$l_3 : \ s = \ln\sqrt{Y_0}, \quad l_4 : \ s = \ln\sqrt{Y_0 - B/X_0}.$$

They intersect at four points, which constitute the "central square" in Figure 2.3. Clearly, the "central square" converges to the point (t_0, s_0) as $\beta \to \frac{\pi}{2}$. We set

$$e^{2t_\pm} = X_0 \pm \sqrt{\frac{BX_0}{Y_0}} \ \text{ and } \ e^{2s_\pm} = Y_0 \pm \sqrt{\frac{BY_0}{X_0}}. \tag{2.3.12}$$

The points (t_+, s_+) and (t_-, s_-) are a pair of vertices of Z in Figure 2.3. They both converge to (t_0, s_0) as $\beta \to \frac{\pi}{2}$. A simple computation shows that the straight line passing through these two vertices, namely the "major axis", is parallel to the straight line $t - s = 0$. This fact makes the "restriction trick" work in the proof of Lemma 4. Moreover, if $s > s_+$ or $s < s_-$, there is a unique $t_c = t_c(s)$ such that $(t_c, s) \in Z$. If $t > t_+$ or $t < t_-$, there is a unique $s_c = s_c(t)$ such that $(t, s_c) \in Z$. These two facts are related to the oscillatory integral estimates in Proposition 2. Indeed, one can see from (2.3.9) that

$$e^{2t_c(s)} = X_0 + \frac{B}{e^{2s} - Y_0}, \quad e^{2s_c(t)} = Y_0 + \frac{B}{e^{2t} - X_0}. \tag{2.3.13}$$

2.3.3 Bounds on the derivatives of $\phi(t,s)$

Given $0 < \epsilon \ll 1$, we denote the ϵ-neighbourhood of Z by

$$Z_\epsilon = \{(t,s) \in \mathbb{R}^2 : \mathrm{dist}((t,s), Z) \leqslant \epsilon\}.$$

In particular, we set $Z_\epsilon = \varnothing$ if $Z = \varnothing$. See Figure 2.4 and 2.5. We decompose the domain $[0,1]^2$ of the phase function into 4 parts:

(1) Non-stationary phase part: $[0,1]^2 \backslash Z_\epsilon$;

(2) Left folds part: $[0,1]^2 \cap \{(t,s) \in Z_\epsilon : s > s_+ + \epsilon \text{ or } s < s_- - \epsilon\}$;

(3) Right folds part: $[0,1]^2 \cap \{(t,s) \in Z_\epsilon : t > t_+ + \epsilon \text{ or } t < t_- - \epsilon\}$;

(4) Young's inequality part: $[0,1]^2 \cap Z_\epsilon \cap ([t_- - \epsilon, t_+ + \epsilon] \times [s_- - \epsilon, s_+ + \epsilon])$.

Lemma 4. *Let $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. Assume that $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are not contained in a common plane. Then we have*

$$\inf |\phi''_{st}| \geqslant \epsilon^2 e^{-CT},$$

where the infimum is taken on $[0,1]^2 \backslash Z_\epsilon$. If $Z \neq \varnothing$, then we have

$$\inf |\phi''_{st}|/|t - t_c(s)| \geqslant \epsilon e^{-CT},$$

where the infimum is taken on $[0,1]^2 \cap \{(t,s) \in Z_\epsilon : s > s_+ + \epsilon \text{ or } s < s_- - \epsilon\}$, and

$$\inf |\phi''_{st}|/|s - s_c(t)| \geqslant \epsilon e^{-CT},$$

where the infimum is taken on $[0,1]^2 \cap \{(t,s) \in Z_\epsilon : t > t_+ + \epsilon \text{ or } t < t_- - \epsilon\}$. The constant $C > 0$ is independent of ϵ and T.

Lemma 5. *For every muti-index $\alpha = (\alpha_1, \alpha_2)$,*

$$\|D^\alpha \phi\|_\infty \leqslant e^{C_\alpha T},$$

where the norm is taken on the unit square $[0,1]^2$. The constant C_α is independent of T.

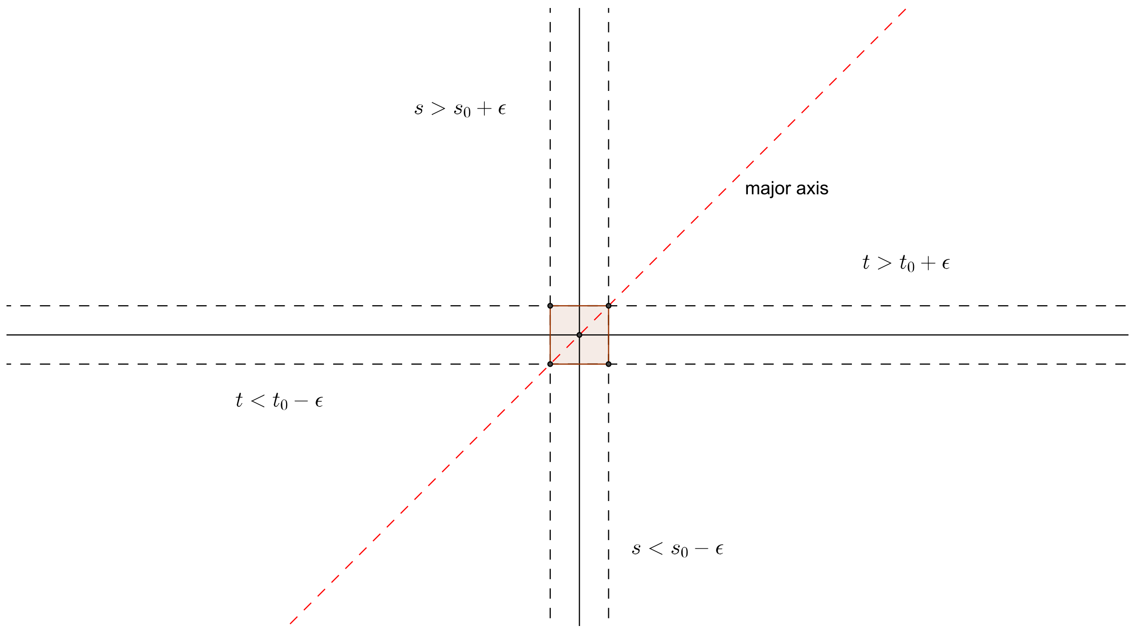

Figure 2.4: Z_ϵ and its decomposition, $\beta = \frac{\pi}{2}$

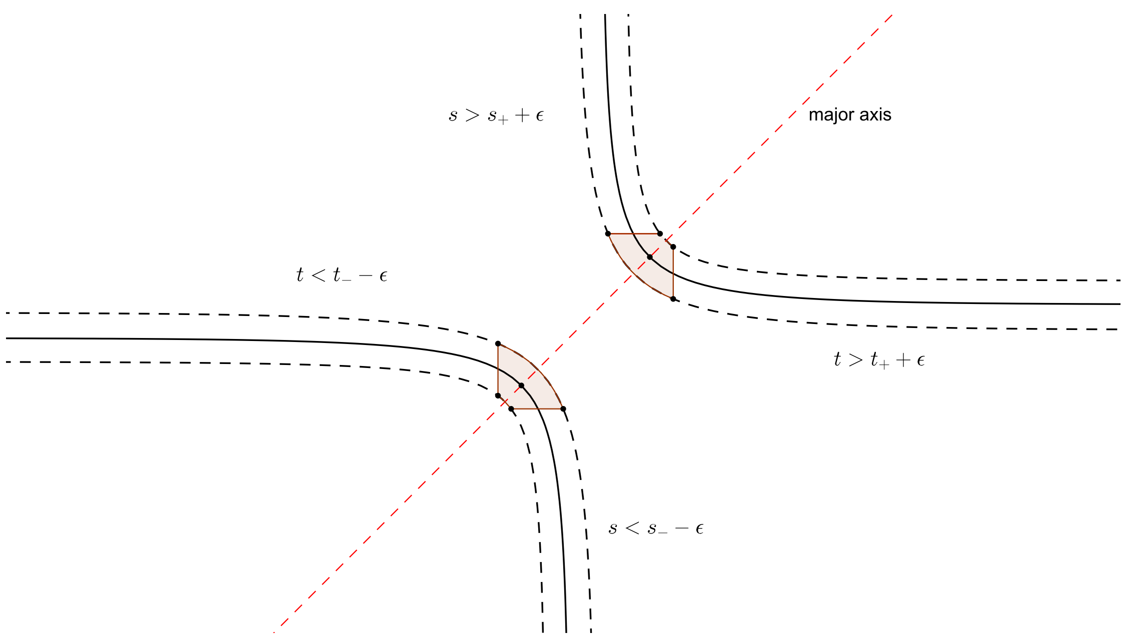

Figure 2.5: Z_ϵ and its decomposition, $\beta \in (0, \frac{\pi}{2})$

2.3.4 End of the proof of the main theorem

We postpone the proof of the lemmas and finish proving Theorem 3. We always use C to denote various positive constants independent of ϵ and T. Recall that there are at most $O(e^{CT})$ summands with $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. We claim that the kernel $K_\lambda^{osc}(t,s)$ of the operator S_λ^{osc} is bounded by $e^{CT}(\epsilon\lambda + \epsilon^{-2}\lambda^{\frac{3}{4}} + \epsilon^{-4}\lambda^{\frac{1}{2}})$. Indeed, one can properly choose some smooth cutoff functions to decompose the domain $[0,1]^2$ and then apply Proposition 2, Lemma 3-5 and Young's inequality to the corresponding parts (1)-(4). Recall that Proposition 2 consists of "non-stationary phase", "left folds" and "right folds". Since the estimate (2.3.5) on the amplitude holds, it is not difficult to see that $\epsilon\lambda$ comes from Young's inequality, $\epsilon^{-2}\lambda^{\frac{3}{4}}$ comes from one-side folds(or stationary phase), and $\epsilon^{-4}\lambda^{\frac{1}{2}}$ comes from non-stationary phase. Then Young's inequality gives

$$\|S_\lambda^{osc}\|_{L^2[0,1]\to L^2[0,1]} \leqslant e^{CT}(\epsilon\lambda + \epsilon^{-2}\lambda^{\frac{3}{4}} + \epsilon^{-4}\lambda^{\frac{1}{2}}). \tag{2.3.14}$$

Taking $T = c\log\lambda$ and $\epsilon = e^{-CT}T^{-1}$, where $c > 0$ is a small constant ($c < (12C)^{-1}$), and combining (2.3.14) with the estimates on S_λ^{tube} (2.2.14) and K_0 (2.2.2), we finish the proof.

2.4 Proof of the bounds of the derivatives

2.4.1 Preliminaries

Before proving the lemmas, we remark that in the Poincaré half-space model

$$\mathrm{T}_R(\tilde{\gamma}) = \{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } z \geqslant \sqrt{x^2 + y^2}/\sqrt{(\cosh R)^2 - 1}\}.$$

See Figure 2.1. Indeed, the distance between $(0, 0, e^t)$ and (x, y, z), $z > 0$, is

$$f(t) = \operatorname{arcosh}\Big(1 + \frac{x^2 + y^2 + (z - e^t)^2}{2ze^t}\Big) = \operatorname{arcosh}\Big(\frac{x^2 + y^2 + z^2 + e^{2t}}{2ze^t}\Big).$$

Setting $f'(t) = 0$ gives $t = \ln\sqrt{x^2 + y^2 + z^2}$, which must be the only minimum point. Thus the distance between (x, y, z) and the infinite geodesic $\tilde{\gamma}$ is

$$\operatorname{dist}((x, y, z), \tilde{\gamma}) = \operatorname{arcosh}(\sqrt{1 + (x/z)^2 + (y/z)^2}).$$

Since $\operatorname{dist}((x, y, z), \tilde{\gamma}) \leqslant R$ in $\mathrm{T}_R(\tilde{\gamma})$, it follows that $z \geqslant \sqrt{x^2 + y^2}/\sqrt{(\cosh R)^2 - 1}$. In the following, we prove Lemma 4 and Lemma 5.

2.4.2 Proof of Lemma 4

First of all, we need to derive some useful results from the condition that $\phi(t,s) \leqslant T$. Namely,

$$(e^{2t}+d_1^2)e^{2s} - 4r(\cosh T)e^t e^s + e^{2t} + d_2^2 \leqslant 0, \ \ t \in [0,1], \ \ s \in I. \tag{2.4.1}$$

Solving the quadratic inequality (2.4.1) about e^s, we have

$$\frac{r}{4\cosh T} \leqslant e^s \leqslant 4r\cosh T. \tag{2.4.2}$$

The discriminant of (2.4.1) has to be nonnegative:

$$16r^2(\cosh T)^2 e^{2t} - 4(e^{2t}+d_1^2)(e^{2t}+d_2^2) \geqslant 0,$$

from which we see that

$$\frac{a}{r} \leqslant 2e\cosh T, \tag{2.4.3}$$

$$d_1 \leqslant 2e\cosh T, \tag{2.4.4}$$

$$r \geqslant \frac{1}{2\cosh T}, \tag{2.4.5}$$

which are similar to the observations in [27, p.21].

Moreover, to get the lower bounds of the derivatives, we need the condition that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. We claim that there exists some constant C independent of T such that

$$\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})} \Rightarrow r \leqslant C\cosh T \text{ or } d_1 \geqslant \frac{1}{C\cosh T}. \tag{2.4.6}$$

Indeed, we are going to prove the contrapositive:

$$r \geqslant C\cosh T \text{ and } d_1 \leqslant \frac{1}{C\cosh T} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}. \tag{2.4.7}$$

We obtain this by showing that under the above assumptions on r and d_1, the segment $\gamma_2(s), s \in [-\ln(4r^{-1}\cosh T), \ln(4r\cosh T)]$ is completely included in $\mathrm{T}_R(\tilde{\gamma})$, which implies $\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ by (2.4.2).

The argument is generalized from [27, p.23]. Solving the polynomial system

$$\begin{cases} z = \sqrt{x^2+y^2}/\sqrt{(\cosh R)^2-1} \\ (x,y,z) = (a + \frac{1-e^{2s}}{1+e^{2s}} r\cos\beta, \frac{1-e^{2s}}{1+e^{2s}} r\sin\beta, \frac{2re^s}{1+e^{2s}}) \end{cases}$$

we can see that

$$\begin{aligned} &\{\gamma_2(s) : s \in \mathbb{R}\} \cap \mathrm{T}_R(\tilde{\gamma}) \\ &= \{\gamma_2(s) : d_1^2 e^{4s} + 2(a^2 + r^2 - 2(\cosh R)^2 r^2) e^{2s} + d_2^2 \leqslant 0\}. \end{aligned} \tag{2.4.8}$$

Note that

$$\begin{cases} r \geqslant C\cosh T \\ d_1 \leqslant (C\cosh T)^{-1} \end{cases} \Rightarrow a/r \leqslant 1 + (C\cosh T)^{-2} \leqslant \sqrt{(\cosh R)^2-1}.$$

This implies

$$\frac{a}{r} \leqslant \sqrt{\frac{(\cosh R)^2-1}{(\cosh R)^2-\cos^2\beta}}\cosh R,$$

which is equivalent to

$$(a^2+r^2-2(\cosh R)^2 r^2)^2 - d_1^2 d_2^2 \geqslant 0.$$

This means that the discriminant of the quadratic polynomial in terms of e^{2s} in (2.4.8) is nonnegative. Thus when $d_1 > 0$, the RHS of (2.4.8) becomes

$$\{\gamma_2(s) : u_- \leqslant e^{2s} \leqslant u_+\}, \tag{2.4.9}$$

where

$$u_\pm = \frac{2(\cosh R)^2 r^2 - r^2 - a^2 \pm \sqrt{(a^2+r^2-2(\cosh R)^2r^2)^2 - d_1^2 d_2^2}}{d_1^2}. \tag{2.4.10}$$

It is easy to see that

$$u_- \leqslant \frac{d_2^2}{2(\cosh R)^2 r^2 - r^2 - a^2} \leqslant \frac{d_2^2}{(\cosh R)^2 r^2} \leqslant \frac{(\cosh R)^2 + 2\cosh R}{(\cosh R)^2}, \tag{2.4.11}$$

$$u_+ \geqslant \frac{(2(\cosh R)^2-1)r^2 - a^2}{d_1^2} \geqslant \frac{(\cosh R)^2 r^2}{d_1^2}. \tag{2.4.12}$$

So if we choose $C = 4\sqrt{\cosh R + 2}/\sqrt{\cosh R}$, we see that

$$d_1 > 0 \text{ and } \begin{cases} r \geqslant C\cosh T \\ d_1 \leqslant (C\cosh T)^{-1} \end{cases} \Rightarrow \begin{cases} u_- \leqslant r^2(4\cosh T)^{-2} \\ u_+ \geqslant (4r\cosh T)^2 \end{cases} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}. \tag{2.4.13}$$

In the easier case $d_1 = 0$, we have $u_+ = +\infty$. Consequently, we obtain (2.4.7), which is equivalent to our claim (2.4.6).

Moreover, we notice that by $\phi \leqslant T$,

$$|\phi_{st}^{''}| \geqslant |\phi_{st}^{''}|\Big(\frac{A}{4re^{s+t}\cosh T}\Big)^2 \geqslant \frac{|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})|}{(\cosh T)^2 r A}. \tag{2.4.14}$$

Now we need to consider two cases: (I) $r \leqslant a\cos\beta$; (II) $r > a\cos\beta$.

Case (I): $\phi_{st}^{''}$ has no zeros and it is not difficult to obtain the lower bound of $|\phi_{st}^{''}|$. Indeed, if $d_1 \geqslant 1$, by (2.4.14) and (2.4.2)-(2.4.3), we get

$$|\phi_{st}^{''}| \geqslant \frac{C(a\cos\beta + r)d_1^2 r^2 (\cosh T)^{-2}}{(\cosh T)^2 r (d_1^2 r^2 (\cosh T)^2)} \geqslant Ce^{-6T}.$$

If $d_1 \leqslant 1$, the claim (2.4.6) is needed. We assume that $r \leqslant C\cosh T$. Then by (2.4.14) and (2.4.2)-(2.4.5), we obtain

$$|\phi_{st}^{''}| \geqslant \frac{C(a\cos\beta + r)e^{2t}}{(\cosh T)^2 r (r^2 (\cosh T)^2)} \geqslant Ce^{-6T}.$$

Otherwise, we assume that $d_1 \geqslant (C\cosh T)^{-1}$. Then similarly we have

$$|\phi_{st}^{''}| \geqslant \frac{C(a\cos\beta + r)d_1^2 r^2 (\cosh T)^{-2}}{(\cosh T)^2 r (r^2 (\cosh T)^2)} \geqslant Ce^{-8T}.$$

Case (II): Since $\phi_{st}^{''}$ has zeros, we prove the lower bound of $|\phi_{st}^{''}|$ on $([0,1] \times I)\backslash Z_\epsilon$ first. The claim (2.4.6) is essential here. However, for technical reasons we only need a slightly weaker but useful version of the claim:

$$\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})} \Rightarrow r \leqslant C(\cosh T)^7 \text{ or } \begin{cases} d_1 \geqslant (C\cosh T)^{-1} \\ r \geqslant C(\cosh T)^7 \end{cases}. \tag{2.4.15}$$

(i) Assume that $r \leqslant C(\cosh T)^7$.

In this case, we use a "restriction trick" to reduce it to a one-variable problem. Let $\delta \in \mathbb{R}$. We restrict $\phi_{st}^{''}(t,s)$ on the straight line $s - t = \delta$ and obtain a uniform lower bound independent of δ.

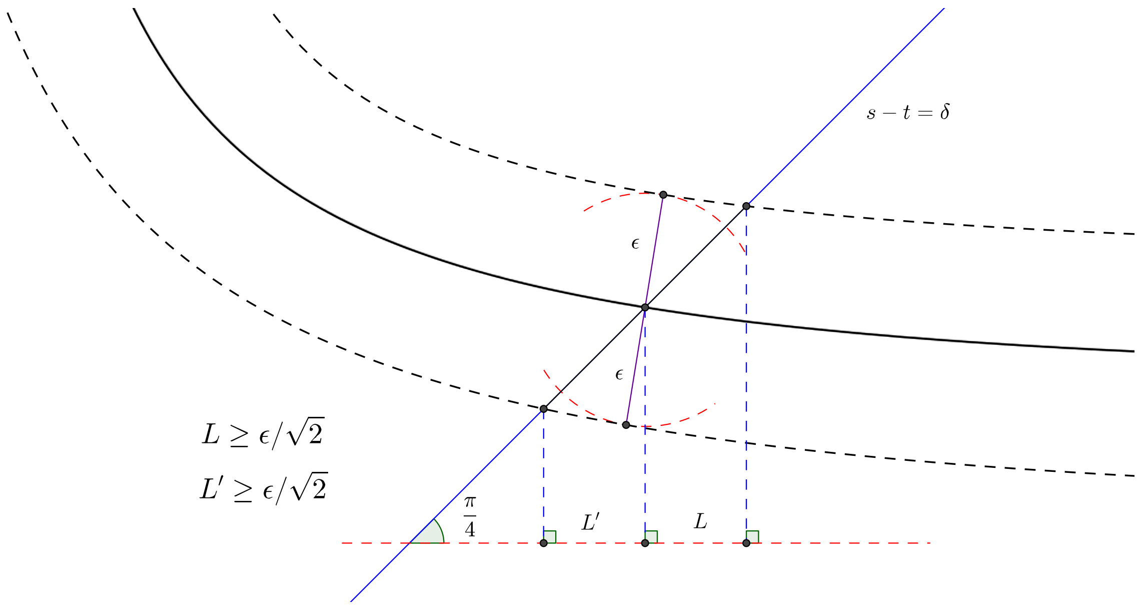

Figure 2.6: Restriction on $s-t=\delta$

Indeed,

$$\begin{aligned}
&|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})| \\
&= (r - a\cos\beta)|e^{2s+2t} - Y_0 e^{2t} - X_0 e^{2s} + d_2^2| \\
&= (r - a\cos\beta)|e^{4t} - (X_0 + Y_0 e^{-2\delta})e^{2t} + d_2^2 e^{-2\delta}|e^{2\delta} \\
&= (r - a\cos\beta)|(e^{2t} - e^{2\tau_-})(e^{2t} - e^{2\tau_+})|e^{2\delta},
\end{aligned}$$

where

$$2e^{2\tau_\pm} = X_0 + Y_0 e^{-2\delta} \pm \sqrt{(X_0 - Y_0 e^{-2\delta})^2 + 4Be^{-2\delta}}.$$

If $r - a\cos\beta \leqslant \frac{r + a\cos\beta}{100} e^{-2\delta}$, then

$$2e^{2\tau_+} \geqslant Y_0 e^{-2\delta} \geqslant 100.$$

But $t \in [0,1]$ implies that

$$|e^{2t} - e^{2\tau_+}| \geqslant \frac{1}{2} e^{2\tau_+} \geqslant \frac{1}{4} Y_0 e^{-2\delta}.$$

Let $Z_{\epsilon,\delta} = \{t \in \mathbb{R} : (t, t+\delta) \in Z_\epsilon\}$. Since the straight line $s - t = \delta$ is parallel to the "major axis" of Z, we have

$$\operatorname{dist}(\tau_\pm, [0,1] \backslash Z_{\epsilon,\delta}) \geqslant \epsilon/\sqrt{2}. \tag{2.4.16}$$

See Figure 2.6. This implies

$$|e^{2t}-e^{2\tau_-}|\geqslant 1-e^{-\epsilon\sqrt{2}}\geqslant \epsilon/10,\ \text{ for } t\in[0,1]\backslash Z_{\epsilon,\delta}.$$

Thus

$$(r-a\text{cos}\beta)|(e^{2t}-e^{2\tau_-})(e^{2t}-e^{2\tau_+})|e^{2\delta}\geqslant \frac{\epsilon}{40}(r+a\text{cos}\beta).$$

If $r-a\text{cos}\beta\geqslant \frac{r+a\text{cos}\beta}{100}e^{-2\delta}$, then we use (2.4.16) again to see that

$$|e^{2t}-e^{2\tau_\pm}|\geqslant 1-e^{-\epsilon\sqrt{2}}\geqslant \epsilon/10,\ \text{ for } t\in[0,1]\backslash Z_{\epsilon,\delta},$$

which gives

$$(r-a\text{cos}\beta)|(e^{2t}-e^{2\tau_-})(e^{2t}-e^{2\tau_+})|e^{2\delta}\geqslant \frac{\epsilon^2}{10000}(r+a\text{cos}\beta).$$

So we can use (2.4.14), (2.4.2)-(2.4.5) and our assumption $r\leqslant C(\text{cosh}T)^7$ to obtain the lower bound of $|\phi_{st}^{''}|$, namely

$$|\phi_{st}^{''}|\geqslant \frac{C\epsilon^2(r+a\text{cos}\beta)}{(\text{cosh}T)^2r(r^2(\text{cosh}T)^4)}\geqslant C\epsilon^2e^{-20T}. \tag{2.4.17}$$

(ii) Assume that $d_1\geqslant(C\text{cosh}T)^{-1}$ and $r\geqslant C(\text{cosh}T)^7$.

If $|r-a\text{cos}\beta|\leqslant 1$, we can use (2.4.2)-(2.4.5) and our assumption to get

$$|(a\text{cos}\beta-r)(e^{2s+2t}+d_2^2)+(a\text{cos}\beta+r)(e^{2t}+d_1^2e^{2s})|\geqslant Cr^3(\text{cosh}T)^{-4},$$

since $(r+a\text{cos}\beta)(d_1^2e^{2s}+e^{2t})\geqslant Cr^3(\text{cosh}T)^{-4}$ and $(r-a\text{cos}\beta)(e^{2s+2t}+d_2^2)\leqslant Cr^2(\text{cosh}T)^2$.

If $|r-a\text{cos}\beta|\geqslant 1$, then $d_1\geqslant|r-a\text{cos}\beta|\geqslant 1$. Thus, $(r+a\text{cos}\beta)(d_1^2e^{2s}+e^{2t})\geqslant Cr^3(\text{cosh}T)^{-2}$ and $(r-a\text{cos}\beta)(e^{2s+2t}+d_2^2)\leqslant Cr^2(\text{cosh}T)^3$, which imply

$$|(a\text{cos}\beta-r)(e^{2s+2t}+d_2^2)+(a\text{cos}\beta+r)(e^{2t}+d_1^2e^{2s})|\geqslant Cr^3(\text{cosh}T)^{-2}.$$

Therefore, we use (2.4.14) and (2.4.2)-(2.4.5) to get

$$|\phi_{st}^{''}|\geqslant \frac{Cr^3(\text{cosh}T)^{-4}}{(\text{cosh}T)^2r(r^2(\text{cosh}T)^4)}\geqslant Ce^{-10T}, \tag{2.4.18}$$

which is better than the bound ϵ^2e^{-CT}. Since the lower bounds in (2.4.17) and (2.4.18) are independent of δ, we finish the proof of the lower bound of $|\phi_{st}^{''}|$ on $([0,1]\times I)\backslash Z_\epsilon$.

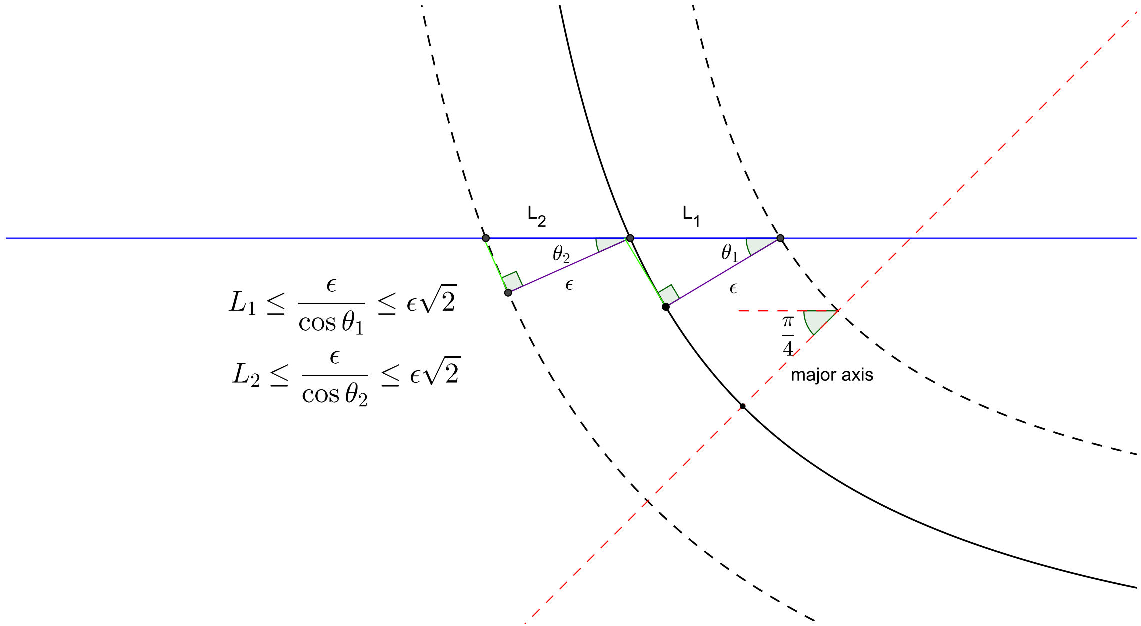

Figure 2.7: $\mathrm{dist}(t_c, [0,1]) \leqslant \epsilon\sqrt{2}$

Now we are ready to give the proof of the lower bounds of $|\phi''_{st}/(t-t_c)|$ and $|\phi''_{st}/(s-s_c)|$. Denote

$$\epsilon_0 = \frac{1}{2}\ln\Big(1+\sqrt{\frac{B}{X_0Y_0}}\Big)+\epsilon.$$

Part 1: Assume that

$$([0,1]\times I)\cap\{(t,s)\in Z_\epsilon : s>s_+ +\epsilon \text{ or } s<s_- -\epsilon\}\neq\varnothing. \tag{2.4.19}$$

We need to obtain the lower bound of $|\phi''_{st}/(t-t_c)|$ on this set. A simple computation using (2.3.10)-(2.3.12) shows that

$$\begin{aligned} s>s_+ +\epsilon \;&\Leftrightarrow\; e^{2s}>Y_0e^{2\epsilon_0},\\ s<s_- -\epsilon \;&\Leftrightarrow\; e^{2s}<(Y_0-B/X_0)e^{-2\epsilon_0}. \end{aligned} \tag{2.4.20}$$

Hence

$$|e^{2s}-Y_0|\geqslant(1-e^{-2\epsilon_0})Y_0.$$

Since the "major axis" of Z is parallel to the straight line $s-t=0$, by our assumption (2.4.19) we have $t_c\in[-\epsilon\sqrt{2}, 1+\epsilon\sqrt{2}]$. See Figure 2.7. Thus,

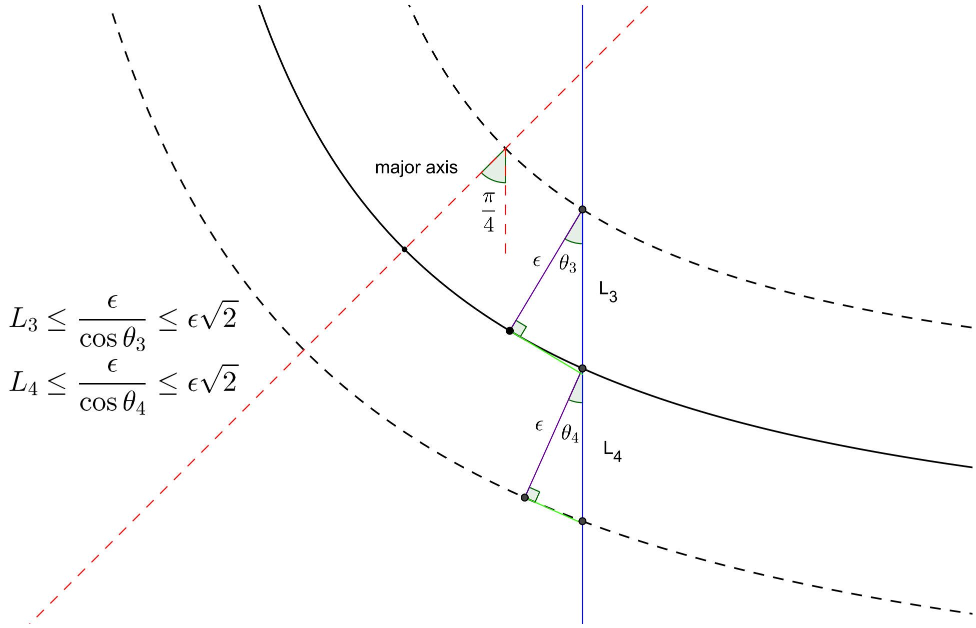

Figure 2.8: $\mathrm{dist}(s_c, I) \leqslant \epsilon\sqrt{2}$

$$
\begin{aligned}
&|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})|/|t - t_c| \\
&= (r - a\cos\beta)\Big|\frac{(e^{2t} - X_0)(e^{2s} - Y_0) - B}{t - t_c}\Big| \\
&= (r - a\cos\beta)\Big|\frac{(e^{2t} - e^{2t_c})(e^{2s} - Y_0)}{t - t_c}\Big| \\
&= (r - a\cos\beta) \cdot 2e^{2t'} \cdot |e^{2s} - Y_0| \\
&\geqslant (r - a\cos\beta) \cdot 2e^{-2\epsilon\sqrt{2}} \cdot (1 - e^{-2\epsilon_0})Y_0 \\
&\geqslant \frac{\epsilon}{100}(r + a\cos\beta),
\end{aligned}
$$

where we use the mean value theorem and $\epsilon_0 \geqslant \epsilon$.

First, we assume that $r \leqslant C(\cosh T)^7$. Then using (2.4.14) and (2.4.3)-(2.4.5), we obtain

$$
\Big|\frac{\phi''_{st}}{t - t_c}\Big| \geqslant \frac{C\epsilon(r + a\cos\beta)}{(\cosh T)^2 r (r^2 (\cosh T)^4)} \geqslant C\epsilon e^{-20T}.
$$

Under the other assumption that "$d_1 \geqslant (C\cosh T)^{-1}$ and $r \geqslant C(\cosh T)^7$", since $|t - t_c| \leqslant 1 + \epsilon\sqrt{2}$ and the lower bound (2.4.18) of $|\phi''_{st}|$ is still applicable here, we get

$$
\Big|\frac{\phi''_{st}}{t - t_c}\Big| \geqslant Ce^{-10T}.
$$

Part 2: Assume that

$$([0,1]\times I)\cap\{(t,s)\in Z_\epsilon : t>t_+ +\epsilon \text{ or } t<t_- -\epsilon\}\neq\varnothing. \tag{2.4.21}$$

We need to get the lower bound of $|\phi''_{st}/(s-s_c)|$ on this set. It is also not difficult to see from (2.3.10)-(2.3.12) that

$$\begin{aligned} t>t_+ +\epsilon \ &\Leftrightarrow\ e^{2t}>X_0e^{2\epsilon_0},\\ t<t_- -\epsilon \ &\Leftrightarrow\ e^{2t}<(X_0-B/Y_0)e^{-2\epsilon_0}. \end{aligned} \tag{2.4.22}$$

Hence

$$|e^{2t}-X_0|\geqslant(1-e^{-2\epsilon_0})\max\{X_0,1\}.$$

If $t>t_+ +\epsilon$, clearly we have $e^{2s_c}\geqslant Y_0$. See Figure 2.3. If $B=0$, we have $e^{2s_c}=Y_0$. If $t<t_- -\epsilon$ and $B>0$, then from (2.3.13) we get

$$\begin{aligned} e^{2s_c}&=Y_0-\frac{B}{X_0-e^{2t}}\geqslant Y_0-\frac{B}{e^{2t+2\epsilon_0}+B/Y_0-e^{2t}}\\ &=Y_0-\frac{B}{e^{2t}(e^{2\epsilon}-1)+e^{2t+2\epsilon}\sqrt{B/(X_0Y_0)}+B/Y_0}\geqslant Y_0-\frac{B}{\sqrt{B/(X_0Y_0)}+B/Y_0}\\ &=Y_0-\frac{X_0Y_0}{\sqrt{X_0Y_0/B}+X_0}\geqslant Y_0-\frac{X_0Y_0}{1+X_0}\\ &=\frac{Y_0}{X_0+1}, \end{aligned}$$

where we use $X_0Y_0/B>1$ from (2.3.11). Since the "major axis" of Z is parallel to the straight line $s-t=0$, by the assumption (2.4.21) we get $\mathrm{dist}(s_c,I)\leqslant\epsilon\sqrt{2}$. See Figure 2.8. Therefore,

$$\begin{aligned} &|(a\cos\beta-r)(e^{2s+2t}+d_2^2)+(a\cos\beta+r)(e^{2t}+d_1^2e^{2s})|/|s-s_c|\\ &=(r-a\cos\beta)\left|\frac{(e^{2t}-X_0)(e^{2s}-Y_0)-B}{s-s_c}\right|\\ &=(r-a\cos\beta)\left|\frac{(e^{2t}-X_0)(e^{2s}-e^{2s_c})}{s-s_c}\right|\\ &=(r-a\cos\beta)|e^{2t}-X_0|\cdot 2e^{2s'}\\ &\geqslant(r-a\cos\beta)|e^{2t}-X_0|\cdot 2e^{2(s_c-1-\epsilon\sqrt{2})}\\ &\geqslant(r-a\cos\beta)(1-e^{-2\epsilon_0})\max\{X_0,1\}\cdot\frac{2Y_0}{X_0+1}e^{-2-2\epsilon\sqrt{2}}\\ &\geqslant\frac{\epsilon}{100}(r+a\cos\beta), \end{aligned}$$

where we use the mean value theorem and $\epsilon_0\geqslant\epsilon$. Then we can obtain the lower bound of $|\phi''_{st}/(s-s_c)|$

in the same way as Part 1. First, under the assumption that $r \leqslant C(\cosh T)^7$, we have

$$\Big|\frac{\phi''_{st}}{s-s_c}\Big| \geqslant \frac{C\epsilon(r+a\cos\beta)}{(\cosh T)^2 r(r^2(\cosh T)^4)} \geqslant C\epsilon e^{-20T}.$$

Under the other assumption that "$d_1 \geqslant (C\cosh T)^{-1}$ and $r \geqslant C(\cosh T)^7$", noting that $|s-s_c| \leqslant 1+\epsilon\sqrt{2}$ and the bound (2.4.18) is still valid here, we get

$$\Big|\frac{\phi''_{st}}{s-s_c}\Big| \geqslant Ce^{-10T}.$$

So far we have finished the proof of all the lower bounds.

2.4.3 Proof of Lemma 5

We only need to prove the upper bounds of mixed derivatives when $\alpha \neq Identity$, since the bounds for pure derivatives are well known in [1], [3]. For convenience, we denote

$$G(t,s) = (a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s}),$$

$$E(t,s) = A^2 - 16r^2 e^{2s+2t}.$$

Recalling the formula (2.3.7), we have $\phi''_{st} = 16re^{2s+2t}GE^{-3/2}$. By induction it is not difficult to see that for any muti-index $\alpha = (\alpha_1, \alpha_2)$

$$D^\alpha\left(\frac{G}{E^\gamma}\right) = E^{-\gamma-|\alpha|} \sum_{0\leqslant|\beta_0|+\cdots+|\beta_{|\alpha|}|\leqslant|\alpha|} C_{\gamma,\alpha,\beta_0,\ldots,\beta_{|\alpha|}} D^{\beta_0}G \cdot D^{\beta_1}E \cdots D^{\beta_{|\alpha|}}E,$$

where $|\alpha| = \alpha_1 + \alpha_2$, and $C_{\gamma,\alpha,\beta_0,\ldots,\beta_{|\alpha|}}$ are constants independent of G and E. Thus,

$$D^\alpha \phi''_{st} = \frac{re^{2s+2t}}{E^{3/2+|\alpha|}} \sum_{0\leqslant|\beta_0|+\cdots+|\beta_{|\alpha|}|\leqslant|\alpha|} C_{\alpha,\beta_0,\ldots,\beta_{|\alpha|}} D^{\beta_0}G \cdot D^{\beta_1}E \cdots D^{\beta_{|\alpha|}}E.$$

From the condition that $\phi(t,s) \geqslant 2$, we have $A \geqslant 4(\cosh 2)re^{s+t}$. Thus,

$$A - 4re^{s+t} \geqslant (4\cosh 2 - 4)re^{s+t}.$$

If $r \geqslant C\cosh T$, then by (2.4.2)-(2.4.5),

$$E \geqslant (A - 4re^{s+t})^2 \geqslant Cr^2e^{2s+2t} \geqslant Cr^4(\cosh T)^{-2},$$

$$|D^\alpha E| \leqslant C_\alpha r^4(\cosh T)^8, \quad |D^\alpha G| \leqslant C_\alpha r^3(\cosh T)^5.$$

Hence,

$$|D^\alpha \phi''_{st}| \leqslant \frac{C_\alpha r(r\cosh T)^2}{(r^4(\cosh T)^{-2})^{3/2+|\alpha|}} r^3(\cosh T)^5(r^4(\cosh T)^8)^{|\alpha|} \leqslant C_\alpha e^{(10|\alpha|+10)T},$$

If $r \leqslant C\cosh T$, then by (2.4.2)-(2.4.5),

$$E \geqslant (A - 4re^{s+t})^2 \geqslant Cr^2e^{2s+2t} \geqslant C(\cosh T)^{-6},$$

$$|D^\alpha E| \leqslant C_\alpha(\cosh T)^{12}, \quad |D^\alpha G| \leqslant C_\alpha(\cosh T)^8.$$

Therefore,

$$|D^\alpha \phi''_{st}| \leqslant \frac{C_\alpha (\cosh T)^5}{((\cosh T)^{-6})^{3/2+|\alpha|}} (\cosh T)^8 ((\cosh T)^{12})^{|\alpha|} \leqslant C_\alpha e^{(18|\alpha|+22)T}.$$

Remark 4. *The condition that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ is essential in the proof of the lower bounds. However, the proof of the upper bounds only needs $2 \leqslant \phi \leqslant T$.*

3

Related Open Problems

1. Generalize Theorem 1 to any smooth curves, or find a counterexample such that the log loss in (1.1.3) cannot be removed. See Remark 1 for a detailed discussion. Also consider the generalizations to higher dimensional manifolds.

2. Obtain log improvement for the L^2 geodesic restriction bound on nonpositively curved manifolds of dimension 3. Theorem 3 only considers the case of constant negative curvature. One of the technical difficulties is that these manifolds may not have sufficiently many totally geodesic submanifolds (see [8, p.458]).

3. Study the optimal restriction estimates on the flat tori and hyperbolic manifolds. These problems are related to the distribution of lattice points on the sphere, Lindelöf Hypothesis for the Riemann zeta function and the quantum unique ergodicity (QUE) conjecture.

Bibliography

[1] P. H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.*, 155(3):249–276, 1977.

[2] M. D. Blair. On logarithmic improvements of critical geodesic restriction bounds in the presence of nonpositive curvature. arXiv:1607.03174, to appear Israel Journal of Mathematics.

[3] M. D. Blair and C. D. Sogge. Concerning toponogov's theorem and logarithmic improvement of estimates of eigenfunctions. arXiv:1510.07726, to appear Journal of Differential Geometry.

[4] M. D. Blair and C. D. Sogge. Refined and microlocal kakeya-nikodym bounds of eigenfunctions in higher dimensions. arXiv:1510.07724.

[5] N. Burq, P. Gérard, and N. Tzvetkov. Restriction of the Laplace-Beltrami eigenfunctions to submanifolds. *Duke Math. J.*, 138:445–486, 2007.

[6] A. Carbery and S. Pérez. Maximal functions and hilbert transforms along variable flat curves. *Mathematical Research Letters*, 6:237–250, 1999.

[7] X. Chen. An improvement on eigenfunction restriction estimates for compact boundaryless Riemannian manifolds with nonpositive sectional curvature. *Trans. Amer. Math. Soc.*, 367:4019–4039, 2015.

[8] X. Chen and C. D. Sogge. A few endpoint geodesic restriction estimates for eigenfunctions. *Comm. Math. Phys.*, 329(2):435–459, 2014.

[9] A. Hassell and M. Tacy. Improvement of eigenfunction estimates on manifolds of nonpositive curvature. *Forum Mathematicum*, 27(3):1435–1451, 2015.

[10] H. Hezari. Quantum ergodicity and L^p norms of restrictions of eigenfunctions. arXiv:1606.08066.

[11] H. Hezari and G. Rivière. L^p norms, nodal sets, and quantum ergodicity. *Advances in Mathematics*, 290:938–966, 2016.

[12] R. Hu. L^p norm estimates of eigenfunctions restricted to submanifolds. *Forum Math.*, 6:1021–1052, 2009.

[13] J. Metcalfe and M. Taylor. Nonlinear waves on 3d hyperbolic space. *Transactions of the American Mathematical Society*, 363(7):3489–3529, 2011.

[14] A. Nagel, J. Vance, S. Wainger, and D. Weinberg. Hilbert transforms for convex curves. *Duke Math. J*, 50(3):735–744, 1983.

[15] A. Nagel and S. Wainger. Hilbert transforms associated with plane curves. *Transactions of the American Mathematical Society*, 223:235–252, 1976.

[16] Y. Pan. Uniform estimates for oscillatory integral operators. *Journal of Functional Analysis*, 100(1):207–220, 1991.

[17] D.H. Phong and E. M. Stein. Hilbert integrals, singular integrals, and radon transforms i. *Acta mathematica*, 157(1):99–157, 1986.

[18] D.H. Phong and E. M. Stein. Radon transforms and torsion. *International Mathematics Research Notices*, 1991(4):49–60, 1991.

[19] A. Reznikov. Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory. arXiv:math/0403437.

[20] F. Ricci and E.M. Stein. Harmonic analysis on nilpotent groups and singular integrals i. oscillatory integrals. *Journal of Functional Analysis*, 73(1):179–194, 1987.

[21] A. Seeger. L^2 estimates for a class of singular oscillatory integrals. *Math. Res. Lett*, 1:65–73, 1994.

[22] C. D. Sogge. Improved critical eigenfunction estimates on manifolds of nonpositive curvature. arXiv:1512.03725.

[23] C. D. Sogge. Concerning the L^p norm of spectral cluster of second-order elliptic operators on compact manifolds. *J. Funct. Anal*, 77:123–138, 1988.

[24] C. D. Sogge. *Hangzhou lectures on eigenfunctions of the Laplacian*, volume 188 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2014.

[25] C. D. Sogge and S. Zelditch. On eigenfunction restriction estimates and L^4-bounds for compact surfaces with nonpositive curvature. In *Advances in Anlysis: The Legacy of Elias M. Stein*, Priceton Mathematical Series, pages 447–461. Princeton University Press, 2014.

[26] M. Taylor. *Partial differential equations II: Qualitative studies of linear equations. Second edition*, volume 116. Springer, New York, 2011.

[27] Y. Xi and C. Zhang. Improved critical eigenfunction restriction estimates on Riemannian surfaces with nonpositive curvature. *Comm. Math. Phys.*, 350(3):1299–1325, 2017.

[28] C. Zhang. Improved critical eigenfunction restriction estimates on riemannian manifolds with constant negative curvature. *Journal of Functional Analysis*, 272(11):4642–4670, 2017.

Curriculum Vitae

Cheng Zhang was born on March 9, 1992 in Guangdong, China. He earned his Bachelor of Science degree in Mathematics and Applied Mathematics from Zhejiang University in 2014. He started his PhD program in mathematics at the Johns Hopkins University from Fall 2014. His dissertation was completed under the guidance of Professor Christopher D. Sogge and defended on June 18, 2019.